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MATHEMATICS.—*The theory of group representations.*[1]

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The modern theory of groups began with what are called permutation groups in the attempts of Lagrange, Ruffini, Vandermonde, and Galois to solve the general algebraic equation of degree greater than 4. Today we usually define a group abstractly by a set of postulates. Suppose we have a finite or infinite set of distinct elements $s, t, u, \cdots$ and a rule of combination of these elements such that two of them may combine to produce a unique element of the set, or such that an element may combine with itself to produce a unique element of the set. We shall call this process of combining "multiplication" and shall write the two combining elements in juxtaposition as is done in ordinary algebraic multiplication; the result of their combination we shall call their "product". We assume the following four postulates satisfied (the first of which has already been included in the above description):

1. The product ts is a unique element u of the set $u = ts$.
2. The associative law holds: $u(ts) = (ut)s$.
3. There exists in the set an element e such $se = s$ for all s of the set. (e is the right identity.)
4. To every element s of the set there corresponds an elements denoted by s^{-1} such that $ss^{-1} = e$. (s^{-1} is the right inverse of s with regard to e.)

Such a set of elements is said to constitute a group. It can be shown that the assumed right identity is unique and that it is also a unique left identity. It can be shown also that the right inverse of an element is unique and is also a unique left inverse for that element, so that we have $se = es = s$ and $s^{-1}s = ss^{-1} = e$.

If it happens that $st = ts$ the group is said to be abelian. We may define the integral powers of an element: $s^2 = s \cdot s$, and by induction $s^n = s \cdot s^{n-1}$. Also $s^{-n} = (s^{-1})^n$ and $s^0 = e$. All the elementary algebraic laws of exponents immediately follow.

If the number of elements in the group is finite the group is a finite group, otherwise infinite. For a finite group the number of elements is the order of the group. Evidently the powers of a single element constitute a group, called a cyclic group. If this group is finite, that is, if $s^n = e$ and $s^m \neq e$, where m is less than n, then n is said to be the order of the element s. A cyclic group of infinite order is said to be a free group.

Elementary examples of groups are numerous. If the rule of combination is ordinary arithmetic multiplication, then the set of all positive rational numbers is a group, in which one is the identity and the inverse of $\frac{p}{q}$ is $\frac{q}{p}$. If the rule of combination is ordinary algebraic multiplication then the four numbers $1, -1, i, -i$ ($i^2 = -1$) constitute a cyclic group of order 4, since the elements are the powers of i.

Sets of nonsingular square matrices constitute groups, the elements of the matrices being numbers of the complex number field and the rule of combination being ordinary matrix multiplication. That is, if

$$A = \begin{pmatrix} a_{11}a_{12}a_{13}\cdots a_{1n} \\ \cdots\cdots\cdots\cdots \\ a_{n1}a_{n2}a_{n3}\cdots a_{nn} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11}b_{12}b_{13}\cdots b_{1n} \\ \cdots\cdots\cdots\cdots \\ b_{n1}b_{n2}b_{n3}\cdots b_{nn} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11}c_{12}c_{13}\cdots c_{1n} \\ \cdots\cdots\cdots\cdots \\ c_{n1}c_{n2}c_{n3}\cdots c_{nn} \end{pmatrix} \text{ and } AB = C$$

then $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$ where $i = 1, 2, \cdots, n$ $j = 1, 2, \cdots, n$

[1] Address of the retiring president of the Philosophical Society of Washington, January 14, 1950.

The identity matrix for a group of matrices each with n rows is the matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and methods of computing the inverse of a nonsingular matrix are well known. The three groups below are simple examples of groups of matrices. The first set of six matrices constitutes a group of order six. Beneath each matrix I have written a letter which may serve to designate the element, since the same group is used as an illustration later. E is the identity-element; A, B and C are of order two, while D and F are of order three. The second set of six matrices likewise constitute a group of order six, ω representing a complex cube root of unity. The same letters as before may be associated with these elements for later identification, E again being the identity, A, B, and C of order 2, and D and F of order 3. The third set of matrices constitute a group of order 8.

$$\underset{E}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{A}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}},$$

$$\underset{B}{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}}, \underset{C}{\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}},$$

$$\underset{D}{\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}}, \underset{F}{\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega \end{pmatrix};$$

where $\omega^3 + \omega + 1 = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Matrices of course arose from "linear transformations" and the groups of matrices are really the groups of the linear transformations which give rise to these matrices. In a mathematical system involving certain variables, $x_1, x_2, \cdots, x_n$, it may be convenient to replace these variables by new variables, $x_1', x_2', \cdots x_n'$, through the medium of a linear transformation:

$$\begin{aligned} x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_i + \cdots + a_{2n}x_n \qquad A \\ \cdots & \quad \cdots\cdots\cdots\cdots\cdots \\ x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n \end{aligned}$$

A second linear transformation may replace the $x_1', x_2', \cdots x_n'$ by $x_1'', x_2'', \cdots x_n''$:

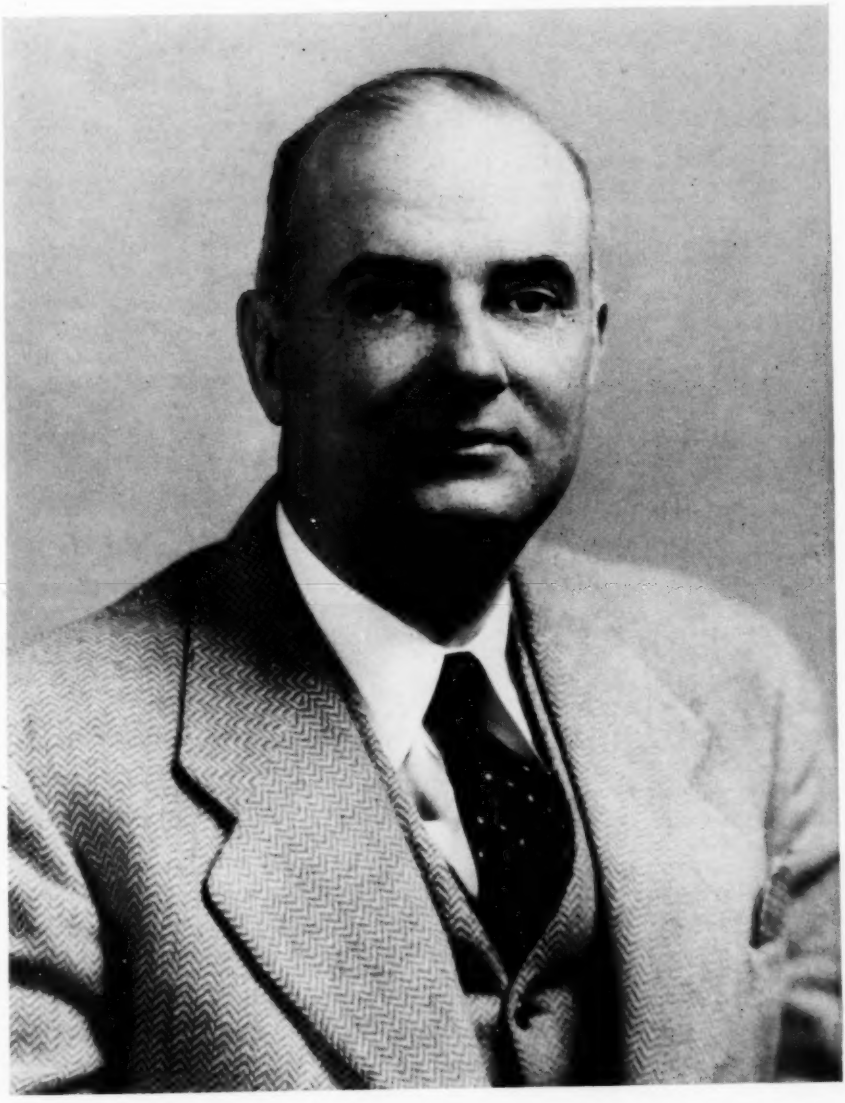
$$\begin{aligned} x_1'' &= b_{11}x_1' + \cdots + b_{1n}x_n' \\ x_2'' &= b_{21}x_1' + \cdots + b_{2n}x_n' \qquad B \\ \cdots & \quad \cdots\cdots\cdots\cdots\cdots \\ x_n'' &= b_{n1}x_1' + \cdots + b_{nn}x_n' \end{aligned}$$

By the "product" of two such linear transformations we mean simply their sequential performance and the result of first operating on the system by the transformation A and then by the transformation B will be the same as if we operated by the single transformation C, where $C = BA$, which replaces the original variables directly by x_1'', x_2'', $\cdots$, x_n''. In practice, however, we generally use just the unaccented variables on the right in a given transformation and simple accented ones on the left, imagining the process to start anew each time.

I said that group theory began with permutation groups in the attempt of mathematicians to solve algebraic equations of higher degree. If we have a set of letters or symbols which appear in a mathematical system, and each of them is replaced by a distinct one of the set then we effect a "permutation" on the letters. This process may be denoted by writing them in any convenient order on one line (the "natural" order if they seem to possess one) and then writing below each letter the one by which it is replaced. Usually large parentheses are used to enclose the array. Thus if the symbols are the letters a, b, c, d, four such permutations are the following:

$$\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}.$$

The first of these represents the identity permutation wherein each letter remains unchanged; in the second permutation a is replaced by b, b by c, c by d, and d by a; in the third a is replaced by c, b by d, c by a,



FRANCIS E. JOHNSTON, President of the Philosophical Society of Washington, 1949.

and d by b; in the last a is replaced by d, b by a, c by b, and d by c. The product of two permutations is their sequential performance; thus the product of the second of the above permutations followed by the third is the last. It is not difficult to verify the fact that the above four permutations satisfy the requirements for a group and hence constitute a permutation group.

Sometimes it is convenient to represent a permutation by a sequence of letters in which each letter is replaced by the letter to its right, the rightmost letter in the "cycle" being replaced by the letter at the beginning of the cycle. If when the cycle is thus "closed" all letters have not been accounted for, a new cycle is begun, and so on until all letters have been taken care of. In this scheme, letters which are replaced by themselves constitute cycles of a single letter and when there is no danger of confusion such cycles are frequently omitted. When a permutation is written in its simplest form as a product of cycles no two cycles will have a common letter, and hence they will be commutative and may be written in any order. Thus

$$\begin{pmatrix} a & b & c & d & e \\ b & e & d & c & a \end{pmatrix} = abe \cdot cd = bea \cdot cd = bea \cdot dc = dc \cdot bea$$

Indeed there are 12 ways to represent the above permutation in the manner under discussion.

Denote by α_1 the number of unary cycles, by α_2 the number of cycles with two letters each, by α_3 the number of cycles with three letters each, etc. Then

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n = n$$

Such a permutation is said to belong to the "class" $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

When a permutation on n letters is written in the 2-row form the letters of the top row may be written in an arbitrary order and those of the lower row may then be written in any one of $n!$ orders; hence there are $n!$ permutations on n letters and the aggregate will constitute a group, the "symmetric" group on n letters. It is not difficult to see (as brought out in the next paragraph) that the number of classes of elements in the symmetric group equals the number of partitions of n with regard to addition into non-negative summands. This is a very old problem in number theory and we tabulate the number of partitions for some values of n:

Value of n	*Number of partitions of n*
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42
16	231
20	627
....	.....
200	3,972,999,029,388.

For example, the partitions of 5 are 5, $4 + 1$, $3 + 2$, $3 + 1 + 1$, $2 + 2 + 1$, $2 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1$, a total of seven.

If we write

$$\begin{aligned} \alpha_1 + \alpha_2 + \cdots + \alpha_n &= \lambda_1 \\ \alpha_2 + \cdots + \alpha_n &= \lambda_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \alpha_n &= \lambda_n \end{aligned}$$

then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq 0$$

and

$$\alpha_1 = \lambda_1 - \lambda_2, \quad \alpha_2 = \lambda_2 - \lambda_3, \cdots,$$
$$\alpha_{n-1} = \lambda_{n-1} - \lambda_n, \quad \alpha_n = \lambda_n$$

A partition of n will thus be in the form $(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)$. For example if n is 10 we have the partition $(3, 2, 2, 1, 1, 1, 0, 0, 0, 0) = (3, 2, 2, 1, 1, 1) = (3, 2^2, 1^3)$. We have omitted unnecessary zeros at the end and also have used exponents to avoid duplication. Here

$$\alpha_1 = 3 - 2 = 1, \quad \alpha_2 = 2 - 2 = 0,$$
$$\alpha_3 = 2 - 1 = 1, \quad \alpha_4 = 1 - 1 = 0$$
$$\alpha_5 = 1 - 1 = 0, \quad \alpha_6 = 1 - 0 = 1,$$
$$\alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10} = 0.$$

Hence the above partition corresponds to a class of elements having the structure: one unary cycle, one ternary cycle, one cycle of six letters. For $n = 3$ and $n = 4$ we list the partitions, the number of elements in the corresponding class, and the actual permutations:

	Partition	Class	Number of permutations	List of permutations
$n = 3$	(1, 1, 1)	$\alpha_1 = 0,\ \alpha_2 = 0\ \alpha_3 = 1$	2	$abc,\ acb$
	(2, 1)	$\alpha_1 = 1,\ \alpha_2 = 1$	3	$ab,\ ac,\ bc$
	(3)	$\alpha_1 = 3$	1	identity
$n = 4$	(1, 1, 1, 1)	$\alpha_1 = \alpha_2 = \alpha_3 = 0,\ \alpha_4 = 1$	6	$abcd,\ adcb,\ abdc,\ acdb$ $acbd,\ adbc$
	(3, 1)	$\alpha_1 = 2,\ \alpha_2 = 1$	6	$ab,\ ac,\ ad,\ bc,\ bd,\ cd$
	(2, 2)	$\alpha_1 = 0,\ \alpha_2 = 2$	3	$ab \cdot cd,\ ac \cdot bd,\ ad \cdot bc$
	(2, 1, 1)	$\alpha_1 = 1,\ \alpha_2 = 0,\ \alpha_3 = 1$	8	$abc,\ acb,\ abd,\ adb$ $acd,\ adc,\ bcd,\ bdc.$
	(4)	$\alpha_1 = 4$	1	identity

Without stopping to advance an argument we note that the number of permutations on n letters in the class $(\alpha_1, \alpha_2, \alpha_3, \cdots)$ is given by

$$\frac{n!}{(1^{\alpha_1} \cdot \alpha_1!)(2^{\alpha_2} \cdot \alpha_2!)(3^{\alpha_3} \cdot \alpha_3!) \cdots}$$

where as usual $0! = 1$. This formula is verifiable in the simple cases above.

A permutation group may be interpreted as a group of matrices. Thus, if we write the permutation in the 2-row form and accent the symbols in the lower row we have a transformation, which is given by its matrix. For example, on three letters the permutation $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2' & x_3' & x_1' \end{pmatrix}$ is essentially the transformation

$$\begin{aligned} x_1' &= x_3 \\ x_2' &= x_1 \\ x_3' &= x_2 \end{aligned}$$

with the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. It will be observed that matrices corresponding to permutations have in each row every element zero except one element, which has the value 1, and likewise in each column they have every element zero except one element which has the value 1.

If $A_1 = E$ (the identity element), $A_2, A_3, \cdots, A_g$ constitute a finite group of matrices and M is a matrix, then the set of matrices $MA_1M^{-1}, MA_2M^{-1}, MA_3M^{-1}, \cdots, MAgM^{-1}$ likewise constitute a group; a group which is *simply isomorphic* with the original group. This means that if $A_iA_j = A_k$ then $(MA_iM^{-1})(MA_jM^{-1}) = MA_kM^{-1}$. The second group is said to be conjugate with the first and is said to be obtained from it by transforming the first group by M. It may happen that the second group will be identical with the first group, perhaps because the individual matrices of the group are unchanged when transformed by M, or because the set of matrices as a whole is identical except for order after it is transformed by M. When this is the case the group is said to be invariant under M.

A matrix $\bar{M}$ is obtained from the matrix M by replacing each of the elements by its conjugate number is the *conjugate* of M. In symbols

$$\bar{M} = (\bar{a}_{ij})$$

A matrix obtained from M by interchanging the roles of the rows and columns of M is called the *transposed* of M and is denoted by M'. That is,

$$M' = (a'_{ij}) \quad \text{where} \quad a'_{ij} = a_{ji}$$

A matrix is said to be *Hermitian* if it equals the transposed of the conjugate of itself, that is H is a Hermitian matrix if

$$H = \bar{H}' \quad \text{or} \quad \bar{a}_{ij} = a_{ji}$$

A matrix is said to be *unitary* if it equals the inverse of the transposed of the conjugate of itself, that is U is unitary if $U = (\bar{U}')^{-1}$. It follows at once that the necessary and sufficient condition that a matrix be unitary is that the relations hold:

$$\sum_{k=1}^{k=n} a_{ki}\bar{a}_{kj} = \delta_{ij}$$

$$i = 1, \cdots, n \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$
$$j = 1, \cdots, n$$

This condition may also be expressed in the equivalent form

$$\sum_{k=1}^{k=n} a_{ik}\bar{a}_{jk} = \delta_{ij}$$

A matrix is said to be *orthogonal* if it equals the inverse of the transposed of itself, that is 0 is orthogonal if

$$0 = (0')^{-1}$$

Evidently for a real matrix the terms orthogonal and unitary mean the same thing.

Suppose we have a group G consisting of the elements $A_1, A_2, \cdots$ and to each element A_i of G we make correspond a non-singular matrix $D(A_i)$ such that if $D(A_i)$ corresponds to A_i and $D(A_j)$ corresponds to A_j then $D(A_jA_i)$ corresponds to A_jA_i. *Then the set of matrices is said to be a representation of* G. We have not said that distinct matrices must correspond to distinct group elements; if that is the case the representation is said to be a faithful representation and the matrices will themselves form a group, a group simply isomorphic with the given group. It is evident that we shall always have a representation in which every group element corresponds to the identity matrix with one row, that is, every element corresponds to the matrix (1). Equally well we might make every element correspond to the identity matrix with two or three or more rows.

Now it may happen that we can find a matrix M such that if we transform simultaneously all the matrices of a representation by it they will all take the form

$$\begin{bmatrix} D^{(1)}(A_i) & 0 & \cdots & 0 \\ 0 & D^{(2)}(A_i) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & D^{(k)}(A_i) \end{bmatrix}$$

If this is true, the original representation is said to be reducible and it has the constituents $D^{(1)}(A_i)$, $D^{(2)}(A_i)$, $\cdots$. Each of these constituents (not necessarily assumed to be all distinct) is also a representation of the given group. If a constituent is such that it cannot be further reduced then it is said to be an irreducible representation; and we assume the transforming matrix so chosen that each $D(A_i)$ is irreducible. Then if the original representation of the group G under discussion is denoted by Γ, Γ is said to be expressed in terms of its irreducible constituents and we write

$$\Gamma(A_i) = D^{(1)}(A_i) \dotplus D^{(2)}(A_i) \dotplus \cdots \dotplus D^{(k)}(A_i).$$

A matrix of Γ is thus the *direct sum* of matrices, one from each of the irreducible representations. Note that this is a different concept from the sum of two matrices. The direct sum means that they are strung out down the main diagonal with zeros elsewhere.

If we have a group $s, t, u, \cdots$ and if we transform an element by one of the group elements we obtain the transform or conjugate, thus tst^{-1} is the transform of s by t. If s and t are commutative, that is if $st = ts$, then $tst^{-1} = s$ and s is said to be invariant under t. In any group an element and all the elements into which it may be transformed by all the elements of the group constitute a *class* of elements. In an abelian group each element is invariant and constitutes a class by itself; hence if the group is finite the number of classes is the order of the group. In the symmetric permutation group all elements with the same cyclic structure are in the same class as noted above. In a non-symmetric group elements with the same cyclic structure may not be all in the same class.

It is a fact that the number of non-equivalent irreducible representations of any finite group equals the number of its classes. Thus the symmetric group on three letters has three classes and three irreducible representations. We show this in the table below:

$$1 \quad (1) \quad (1) \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E$$

$$acb \quad (1) \quad (1) \quad \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad D$$

$$abc \quad (1) \quad (1) \quad \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \quad F$$

$$ab \quad (1) \quad (-1) \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A$$

$$bc \quad (1) \quad (-1) \quad \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \quad B$$

$$ac \quad (1) \quad (-1) \quad \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \quad C$$

The first of these representations is of dimension one and is the representation in which every element corresponds to the identity matrix. The second is of dimension one also and three of the elements correspond to the identity matrix, three to the matrix (-1). The third is of dimension 2 and is a faithful representation

It is a fact that every finite group may be made simply isomorphic to a permutation group, indeed in many ways. In particular it may have a faithful representation as a *regular* permutation group, that is a group in which every letter is replaced by every other letter of the group by one and only one permutation, which means that no permutation (except the identity) leaves any letter invariant. In this procedure the number of letters is the order of the group. Thus the symmetric group on three letters is of order 6, and hence may have a faithful representation as a regular permutation group on six letters. It is also a fact that there always exists a matrix which will transform this representation into the sum of its irreducible constituents and that in this reduced form every irreducible representation will appear and that the number of times it appears is equal to its dimension. We saw above that the symmetric group on three letters (whose order is six) had exactly three irreducible representations, two of dimension one and one of dimension two. In the reduced form the representation of dimension two should therefore appear twice, as is indeed the case. From this we conclude that the sum of the squares of the dimensions of the irreducible representations should equal the order of the group. That is, in the case under consideration: $1^2 + 1^2 + 2^2 = 6$. To give the notion a little concrete reality we exhibit the above mentioned symmetric group and the transforming matrix and the reduced form.

As a regular permutation group the group is as follows: E = identity, $A = x_1x_4 \cdot x_2x_6 \cdot x_3x_5$, $B = x_1x_5 \cdot x_2x_4 \cdot x_3x_6$, $C = x_1x_6 \cdot x_2x_5 \cdot x_3x_4$, $D = x_1x_3x_2 \cdot x_4x_6x_5$, $F = x_1x_2x_3 \cdot x_4x_5x_6$. Interpreted as a group of linear transformations (whose matrices give the regular representation) we have:

E = identity

$$A = \begin{Bmatrix} x_1' = \cdots\cdots x_4\cdots \\ x_2' = \cdots\cdots x_6\cdots \\ x_3' = \cdots\cdots x_5\cdots \\ x_4' = x_1\cdots\cdots \\ x_5' = \cdots x_3\cdots\cdots \\ x_6' = \cdots x_2\cdots\cdots \end{Bmatrix}$$

$$B = \begin{Bmatrix} x_1' = \cdots\cdots x_5\cdots \\ x_2' = \cdots\cdots x_4\cdots \\ x_3' = \cdots\cdots x_6\cdots \\ x_4' = \cdots x_3\cdots\cdots \\ x_5' = x_1\cdots\cdots \\ x_6' = \cdots x_2\cdots\cdots \end{Bmatrix} \quad C = \begin{Bmatrix} x_1' = \cdots\cdots x_6\cdots \\ x_2' = \cdots\cdots x_5\cdots \\ x_3' = \cdots\cdots x_4\cdots \\ x_4' = \cdots x_3\cdots\cdots \\ x_5' = \cdots x_2\cdots\cdots \\ x_6' = \cdots x_1\cdots\cdots \end{Bmatrix}$$

$$D = \begin{Bmatrix} x_1' = \cdots x_3\cdots\cdots \\ x_2' = \cdots\cdots x_4\cdots \\ x_3' = x_1\cdots\cdots\cdots \\ x_4' = \cdots\cdots x_5\cdots \\ x_5' = \cdots\cdots x_6\cdots \\ x_6' = \cdots\cdots x_4\cdots \end{Bmatrix} \quad F = \begin{Bmatrix} x_1' = \cdots x_3\cdots\cdots \\ x_2' = x_1\cdots\cdots\cdots \\ x_3' = \cdots x_2\cdots\cdots \\ x_4' = \cdots\cdots x_6\cdots \\ x_5' = \cdots\cdots x_4\cdots \\ x_6' = \cdots\cdots x_5\cdots \end{Bmatrix}$$

The matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega^2 & \omega \\ 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & 0 & 0 & 0 \end{pmatrix}$$

with the inverse

$$T^{-1} = \frac{1}{6} \begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 2 \\ 1 & 1 & 2\omega^2 & 0 & 0 & 2\omega \\ 1 & 1 & 2\omega & 0 & 0 & 2\omega^2 \\ 1 & -1 & 0 & 2 & 2 & 0 \\ 1 & -1 & 0 & 2\omega & 2\omega^2 & 0 \\ 1 & -1 & 0 & 2\omega^2 & 2\omega & 0 \end{pmatrix}$$

will transform the regular representation in this fashion

$$TET^{-1} = E', TAT^{-1} = A', \cdots, TFT^{-1} = F'$$

where the matrices for the primed representation are as follows:

$$E' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \end{pmatrix} \quad C' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \\ 0 & 0 & 0 & 0 & \omega & 0 \end{pmatrix}$$

$$D' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix} \quad F' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega \end{pmatrix}$$

It is a simple matter to verify this result by a little computation. Indeed it is only necessary to verify it for two of the "generators," say A and B.

We note a few simple and interesting facts about the representations of a finite group. In the first place there always exists a matrix which will transform all the matrices of a representation of a finite group into unitary matrices. This is also true of many infinite groups. All the representations we have noted so far have been unitary representations and we shall always assume that our representations are unitary. When two equivalent representations are unitary, the transforming matrix which carries one representation into the other may be chosen to be unitary. For convenience the transforming matrix T above was not so chosen, but it could have been.

If we have an *irreducible* representation of dimension l_1 of a group of order g and we construct the l_1^2 vectors in the g-dimensional space of the group elements, then we have a set of orthogonal unitary vectors (except that the "Hermitian length" of the vectors is $\sqrt{g/l_1}$). Thus for the two dimensional representation of the symmetric group on three letters we have the four vectors $(1, \omega, \omega^2, 0, 0, 0)$, $(0, 0, 0, 1, \omega, \omega^2)$, $(0, 0, 0, 1, \omega^2, \omega)$, $(1, \omega^2, \omega, 0, 0, 0)$. If we add to this system l_2^2, l_3^2 etc. vectors corresponding to other non-equivalent irreducible representations in the g-dimensional space of the group elements we then have a system of $l_1^2 + l_2^2 + l_3^2 + \cdots$ unitary orthogonal vectors, with the same assumptions as to the Hermitian length of the vectors. Thus in the case of the three representations of the symmetric group above we should have to add to the four vectors above the two vectors $(1, 1, 1, 1, 1, 1)$, and $(1, 1, 1, -1, -1, -1)$. These last two vectors each correspond to representations of dimension one so that the Hermitian length in each case would be $\sqrt{6}$. We write the six vectors together so that the relation is more clearly discernable.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 0 & 0 & 0 & 1 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & 0 & 0 & 0 \end{pmatrix}$$

If $D(R)$ is the matrix corresponding to the element R in any particular representation Γ of the group G then the sum of the elements in the main diagonal of $D(R)$—that is, the *trace* of $D(R)$—is the *characteristic* of R for that particular representation; it is usually denoted by the symbol $\chi(R)$. When R runs through all the group elements there results a set of g numbers, which may be interpreted as a vector in the g dimensional space of the group elements. As is well known the trace of a matrix is invariant under transformations. Hence it is a fact that every element in the same class will have the same characteristic in any particular representation, and it is customary therefore in general to write $\chi(C_j)$ in lieu of $\chi(R)$ where C_j represents the class of elements to which R belongs. If therefore G contains k classes, $C_1, C_2, \cdots, C_k$, containing $g_1, g_2, \cdots, g_k$ elements respectively, where of course $g_1 + g_2 + \cdots + g_k = g$, we shall have for a particular representation the k numbers $\chi(C_1), \chi(C_2), \cdots, \chi(C_k)$, and these may be interpreted as a vector in the k dimensional space of the classes. This vector is sometimes called a character of G. G would thus have a character for each representation, but we shall reserve the term for what are sometimes called simple characters, that is the characters of the k irreducible representations $D^{(1)}(A)$, $D^{(2)}(A)$, $\cdots$, $D^{(k)}(A)$. Characters of other representations can be called compound characters or generalized characters if it is necessary to refer to them. It is evident that two equivalent irreducible representations have the same character. We shall use a superscript to denote the particular representation which gives rise to a character and a subscript to denote the particular class to which a characteristic belongs, thus $\chi^i(C_j)$ represents the characteristic of each

element in the class C_j for the representation $D^{(i)}(A)$.

Evidently if we are in possession of all the irreducible representations of G the process of writing down a table of characters, essentially a matrix of k columns, will be a trivial one. To obtain the irreducible representations may except for elementary cases be a tedious process. However, there are ways by which the characters may be obtained directly. These too may become laborious if we proceed to too complicated groups. Methods of simplifying and improving such processes will delight and inspire or vex and impede the pure mathematician, as the case may be. However, the physicist frequently makes use of only those characters which are most readily obtainable and so his case is not a hopeless one. For the symmetric group in particular the necessary characters are easily obtainable.

We make a few general observations concerning the characters of the symmetric group. We observe first that the permutations of the symmetric group may be divided into two equal sub-sets, the even permutations and the odd permutations. The even permutations are those which leave invariant the alternating function

$$P = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \cdots (x_1 - x_n)$$
$$(x_2 - x_3)(x_2 - x_4) \cdots (x_2 - x_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(x_{n-1} - x_n)$$

The odd permutations are those which reverse the sign of P. Evidently the even permutations form a group, the alternating group on n letters. We have noted that there is a representation of the symmetric group corresponding to each partition of n. Let us make a diagram corresponding to a partition, say the partition $(\lambda_1, \lambda_2, \lambda_3, \cdots)$. Let us put λ_1 dots equally spaced in a horizontal row. Immediately below let us place λ_2 dots the leftmost one immediately below the leftmost one in the top row and put each dot below one in the line above. We place λ_3 dots in a similar manner in the third row and so on. We have thus constructed a sort of triangular matrix of dots, and to every partition there will correspond such a diagram. If we transpose one of these matrices about its main diagonal, we will change it into another one of the diagrams. Two partitions whose diagrams are so related are called associated partitions. Thus the partition (5, 2) of 7 with the diagram ····· is associated with the
··
partition (2, 2, 1, 1, 1) of 7 which has ·· as
··
·
·
·
its diagram. In case the dot matrix is symmetric about the main diagonal, the partition is said to be self-associated. Thus (4, 1, 1, 1) with the diagram ···· is a self-associated
·
·
·
partition of 7. The representations corresponding to associated partitions are associated representations; a characteristic of an even class in a representation is the same as the characteristic of that same class in the associated representation, while a characteristic of an odd class is the negative of the characteristic in the associated representation. We thus see that when we know the character of a representation, we at once know the character of the associated representation. We observe also that it follows that the characteristics of the odd classes in a self-associated representation are all zero.

We observe that we always let C_1 represent the class consisting of the identity alone and that hence $\chi^i(C_1)$ is always the dimension of the representation $D^{(i)}(A)$, being the trace of the identity matrix. We tabulate below the tables of characters for the symmetric group on four symbols and on five symbols:

Partition / Class	(4) †	(3, 1) ††	(2, 2) *	(2, 1, 1) ††	(1, 1, 1, 1) †	Number of elements in class	Partition giving rise to class
(4, 0, 0, 0)+	1	3	2	3	1	1	(4)
(2, 1, 0, 0)-	1	1	0	−1	−1	6	(3, 1)
(1, 0, 1, 0)+	1	0	−1	0	1	8	(2, 1, 1)
(0, 2, 0, 0)+	1	−1	2	−1	1	3	(2, 2)
(0, 0, 0, 1)-	1	−1	0	1	−1	6	(1, 1, 1, 1)

† associated partitions.
†† associated partitions.
* self associated partition.

Class \ Partition	(5) †	(4, 1) ††	(3, 2) †††	(3, 1, 1) *	(2, 2, 1) †††	(2, 1, 1, 1) ††	(1, 1, 1, 1, 1) †	No. of elements in class	Partition giving rise to class
$(5, 0, 0, 0, 0)_+$	1	4	5	6	5	4	1	1	(5)
$(3, 1, 0, 0, 0)_-$	1	2	1	0	−1	−2	−1	10	(4, 1)
$(2, 0, 1, 0, 0)_+$	1	1	−1	0	−1	1	1	20	(3, 1, 1)
$(1, 2, 0, 0, 0)_+$	1	0	1	−2	1	0	1	15	(3, 2)
$(1, 0, 0, 1, 0)_-$	1	0	−1	0	1	0	−1	30	(2, 1, 1, 1)
$(0, 1, 1, 0, 0)_-$	1	−1	1	0	−1	1	−1	20	(2, 2, 1)
$(0, 0, 0, 0, 1)_+$	1	−1	0	1	0	−1	1	24	(1, 1, 1, 1, 1)

† associated partitions.
†† associated partitions.
††† associated partitions.
* self-associate partition.

We observe that if g is the order of the group and g_i is the number of elements in the Class C_i and that if we multiply the characteristic in each box by $\sqrt{\frac{g_i}{g}}$ then the resulting matrix will be a unitary matrix—that is we shall have

$$\sum_{k=1}^{k=n} a_{ki}\,\bar{a}_{kj} = \delta_{ij}$$

and

$$\sum_{k=1}^{k=n} a_{ik}\,\bar{a}_{jk} = \delta_{ij}.$$

So far we have discussed the symmetric group, and that is what primarily interests us here. It so happens that the characteristics here are all real so that the unitary property of the matrix of the characters does not become evident. By way of contrast and completeness we give the table of characters for some simple nonsymmetric groups, in particular the "octic" group (a group of order eight which plays an important role in the theory of the solution of the fourth degree equation), the alternating group on four letters, the simple group of order 168, and the non-cyclic group of order 21. As a permutation group the octic group may be represented as follows.

identity, $ac\cdot bd$, $abcd$, $adcb$, $ab\cdot cd$, $ad\cdot bc$, ac, bd.
C_1 C_2 C_3 C_3 C_4 C_4 C_5 C_5

There are thus five classes and the classes may be enumerated as shown above where the class to which the element belongs is indicated immediately below the class. The table of characters follows:

Class \ Representation	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Number of elements in class
C_1	1	2	1	1	1	1
C_2	1	−2	1	1	1	1
C_3	1	0	−1	−1	1	2
C_4	1	0	−1	1	−1	2
C_5	1	0	1	−1	−1	2

Class \ Representation	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Number of elements in class
identity	1	6	7	8	3	3	1
elements of order 2	1	2	−1	0	−1	−1	21
elements of order 4	1	0	−1	0	1	1	42
elements of order 3	1	0	1	−1	0	0	56
24 elements of order 7	1	−1	0	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$	24
inverses of elements in above class	1	−1	0	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$	24

The alternating group on 4 letters (also known as the tetrahadral group) is as follows:

iden;	$ab\cdot cd$,	$ac\cdot bd$,	$ad\cdot bc$;	abc,	acd,	adb,	bdc;	acb,	adc,	abd,	bcd
C_1	C_2	C_2	C_2	C_3	C_3	C_3	C_3	C_4	C_4	C_4	C_4

The table of characters:

Class \ Representation	(4)	(3, 1)	(2, 2)	(2, 2)′	No. of elements in class
C_1	1	3	1	1	1
C_2	1	−1	1	1	3
C_3	1	0	ω	ω^2	4
C_4	1	0	ω^2	ω	4

The simple group of order 168 is tabulated at the foot of page 125.

The noncyclic group of order 21:

Class \ Representation	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Number of Elements in Class
identity	1	1	1	3	3	1
seven elements of order 3	1	ω	ω^2	0	0	7
inverses of above elements	1	ω^2	ω	0	0	7
three elements of order 7	1	1	1	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$	3
inverses of above elements	1	1	1	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$	3

So far we have dealt with finite groups. We consider infinite groups, in particular continuous groups. The set of all nonsingular n-rowed matrices with elements in the complex number field constitutes a group, the full linear group of dimension n. We shall be concerned with certain subgroups thereof. In particular we shall consider those groups in which the elements are continuous functions of one or more parameters, whose domain of variability may be disconnected or simply or multiply connected. If the domain is connected, the group is a simply continuous group, otherwise a mixed continuous group. We assume that the elements of the matrices possess derivatives of all necessary orders with regard to the parameters.

Moreover, we shall consider groups whose matrices are unitary—the unitary groups; and also we shall consider groups whose determinants are all 1—the unimodular groups. If we consider groups whose matrices are both, we have the unimodular unitary groups.

Ever since we studied elementary analytic geometry we have been familiar with the rotations of the Cartesian plane about the origin. This group is the two dimensional pure rotation group. The matrices are all real orthogonal (hence unitary) and have determinant one. Hence the pure rotation group is real, unimodular and unitary. It is a fact that the real unimodular orthogonal group of dimension n will always have exactly $\frac{1}{2}n(n-1)$ independent parameters; hence in the present case a single parameter. (If we add the real orthogonal transformations of determinant -1, we have the rotation reflection group and we now have two parameters.) The familiar transformation

$$x' = x\cos\phi - y\sin\phi$$

$$y' = x\sin\phi + y\cos\phi$$

with the matrix $\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$

gives us the rotation group the parameter being ϕ where $-\pi < \phi \leq \pi$ a simply connected domain. There is only one parameter and it appears additively in the group, that is, if $\{\phi\}$ represents the element whose parameter is ϕ then $\{\phi + \phi'\} = \{\phi\}\{\phi'\}$. The parameter appears additively and the group is abelian, every element thus being in a class by itself. We seek the non-equivalent irreducible representations of the two-dimensional pure rotation group. They are $(e^{im\phi})$ where ϕ is the parameter and m is a rational integer, $m = \cdots, -2, -1, 0, 1, 2, \cdots$. This time we are concerned with the actual representations rather than merely with the characteristics, though of course the characteristic can immediately be read off.

If we extend the pure rotation group to include also the matrices of determinant -1 we have the entire "rotation-reflection" group, the aggregate of all real orthogonal

matrices. We may get them by adding to the above matrices the matrices

$$\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

We now have two parameters, the continuous parameter ϕ as before and the discrete parameter d which may take either of the values 1 or -1; and the set of matrices may be included in the formula

$$\{\phi, d\} = \begin{pmatrix} d\cos\phi & -d\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

The group is no longer abelian—the set of matrices which constitute the pure rotations no longer consists of matrices each of which is a class by itself, for $\{\phi, 1\}$ and $\{-\phi, 1\}$ now constitute a class. Also all elements

$$\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

are in a class. The group is now a mixed continuous group since the domain of variability of the parameters is no longer connected.

We have of course the trivial representation of the rotation reflection group in which every element corresponds to the matrix (1) and also the one-dimensional representation in which the matrices $\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ correspond to (1) while the matrices $\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ correspond to (-1). The other irreducible representations are two dimensional and in them we have the correspondence

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \leftrightarrow \begin{pmatrix} e^{-im\phi} & 0 \\ 0 & e^{im\phi} \end{pmatrix}$$

$$\begin{pmatrix} -\cos\phi & \sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & e^{im\phi} \\ e^{-im\phi} & 0 \end{pmatrix}$$

for all properly positive integral values of m.

We now consider the irreducible representations of the three dimensional pure rotation group and of the three dimensional rotation-reflection group.

As we have observed the three dimensional pure rotation group (real orthogonal matrices with determinant $+1$) has $\frac{1}{2}3(3-1) = 3$ parameters. There is a very close relation between the representation of this group and the representations of the *unimodular unitary* group in two dimensions. From the latter we may get the former; it is also true that from the latter we may get what are called the "ambiguous" representations of the pure rotation group –they are not truly representations but they play an important role in the theory of the spin of the electron. We have an irreducible representation of the pure rotation group for each zero or positive integral value of j as given by the expression below. The rotation is here given by its Eulerean angle $\{\alpha, \beta, \gamma\}$ the three parameters of the group. The dimension is $2j+1$, and μ' and μ take on the $2j+1$ values $-j$, $-j+1, \cdots, -2, -1, 0, 1, 2, \cdots, j-1, j$. μ' gives the row of the matrix, μ the column, so that the element in the upper left-hand corner is in the position $-j, -j$; the element in the upper right corner is in the position $-j, j$. The representation is denoted by $D^{(j)}\{\alpha, \beta, \gamma\}$. The element in the μ', μ position is given by the expression

$$D^{(j)}(\{\alpha\beta\gamma\})_{\mu'\mu} = \sum_x (-1)^x \cdot \frac{\sqrt{(j+\mu)!(j-\mu)!(j+\mu')!(j-\mu')!}}{(j-\mu'-x)!(j+\mu-x)!x!(x+\mu'-\mu)!} \cdot e^{i\mu'\alpha}(\cos\tfrac{1}{2}\beta)^{2j+\mu-\mu'-2x}(\sin\tfrac{1}{2}\beta)^{2x+\mu'-\mu}e^{i\mu\gamma}$$

where

$$\text{larger of } \begin{Bmatrix} 0 \\ \mu-\mu' \end{Bmatrix} \leq x \leq \text{smaller of } \begin{Bmatrix} j-\mu' \\ j+\mu \end{Bmatrix}$$

For $j=0$ this reduces to the trivial representation in which every rotation corresponds to the one-dimensional matrix (1). For $j=1$ this representation is three dimensional and the matrix corresponding to the rotation $\{\alpha, \beta, \gamma\}$ is given below, the half angles having been changed to integral angles through elementary trigonometric identities:

$$\begin{bmatrix} e^{-i\alpha}\dfrac{1+\cos\beta}{2}e^{-i\gamma} & -e^{-i\alpha}\dfrac{\sin\beta}{\sqrt{2}} & e^{-i\alpha}\dfrac{1-\cos\beta}{2}e^{i\gamma} \\ \dfrac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\dfrac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ e^{i\alpha}\dfrac{1-\cos\beta}{2}e^{-i\gamma} & e^{i\alpha}\dfrac{\sin\beta}{\sqrt{2}} & e^{i\alpha}\dfrac{1+\cos\beta}{2}e^{i\gamma} \end{bmatrix}$$

For half integral values of j we have the ambiguous representation the form of the general element of which is just like that for integral values of j except that before each element there appears the $\pm$ sign. This does not mean that the sign has just not yet been chosen and will be chosen to suit our purposes. It is impossible to do this. The sign must be left ambiguous and we pick whichever one will fit into the particular object we have in mind. This time μ' and μ take the half-integral values $-j,\ -j+1,\ \cdots,\ -\frac{1}{2},\ \frac{1}{2},\ \cdots,\ j$. Thus if j is $\frac{1}{2}$ the representation is of dimension $2j+1=2$ and the four positions of the representing matrices are as shown: $-\frac{1}{2},\ -\frac{1}{2}$ $\quad -\frac{1}{2},\ \frac{1}{2}$ $\quad \frac{1}{2},\ -\frac{1}{2}$ $\quad \frac{1}{2},\ \frac{1}{2}$ The actual representation when $j=\frac{1}{2}$ is shown:

$$\begin{bmatrix} e^{-\frac{1}{2}i\alpha}\cos\dfrac{\beta}{2}e^{-\frac{1}{2}i\gamma} & -e^{-\frac{1}{2}i\alpha}\sin\dfrac{\beta}{2}e^{\frac{1}{2}i\gamma} \\ e^{\frac{1}{2}i\alpha}\sin\dfrac{\beta}{2}e^{-\frac{1}{2}i\gamma} & e^{\frac{1}{2}i\alpha}\cos\dfrac{\beta}{2}e^{\frac{1}{2}i\gamma} \end{bmatrix}$$

The three dimensional rotation reflection group may be thought of as the direct product of the three dimensional pure rotation group and the three dimensional reflection group consisting of the two matrices

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \text{ and } -I=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}$$

(*A* group G is the direct product of H and K if H and K are subgroups of G, have no common element except the identity, if every element of H is commutative with every element of K and if every element of G equals the product of an element of H and an element of K). If G is the direct product of H and K, that is, in symbols, if $G=H\times K$, and if $D_h^{(i)}(s)$ is an irreducible representation of H and $D_k^{(j)}(t)$ is an irreducigle representation of K then an irreducible representation of G is obtained by making the element st of G correspond to the matrix $D_h^{(i)}(s)\times D_k^{(j)}(t)$, that is to the Kronecker product of the matrices, one from the representation of H and the other from the representation of K. It is thus seen that the number of irreducible representations of G equals the product of the number of irreducible representations of H multiplied by the number of irreducible representations of K (provided of course that in each case the number of representations is finite). Since H and K are commutative it might seem that we could obtain another representation by taking the Kronecker products in the reverse order, but it is a fact that the representation $D_h^{(i)}(s)\times D_k^{(j)}(t)$ is equivalent to (may be transformed into) the representation $D_k^{(j)}(t)\times D_h^{(i)}(s)$.

Evidently the three dimensional reflection group has two non-equivalent irreducuble representations, the trivial one in which both I and $-I$ correspond to the matrix (1), and the one in which I corresponds to the matrix (1) and $-I$ corresponds to the matrix (-1). Thus the rotation reflection group has a representation in which each element of the rotation group corresponds to the matrix to which it corresponds in a given representation of the rotation group, while the negative of each element of the rotation group also corresponds to the matrix to which the original element corresponds in the representation of the rotation group. Also the rotation reflection group has a representation in which each element of the rotation group corresponds to the matrix to which it corresponds in a given representation of the rotation group, while the negative of each element of the rotation group corresponds to the negative of the matrix to which the element of the rotation group corresponds. Thus are obtained from the irreducible representations of the rotation group the irreducible representations of the rotation reflection group.

It is a fact that the Schrödinger equation $H\Psi=E\cdot\Psi$ associated with an atomic system consisting of a nucleus and n electrons in their orbits about the nucleus is "invariant under a group" consisting of the direct product of the symmetric group on n elements and the rotation group and the reflection group. It is a fact that the energy levels for which this equation has a solution are the "eigen-

values" of the system and that associated with each eigenvalue is a set of linearly independent "eigenfunctions" which gives the probabilities for a certain state of the atom. The group-theoretic properties discussed above, in particular the representations of the rotation group and of the reflection group and the characters of the symmetric group, have been employed to shed light on this important question of the state of the atom; but the complete story of that application is too long to be told here.

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MATHEMATICS.—*On an equation of Neményi and Truesdell.* D. S. MITRINOVITCH, Institute of Mathematics, Skopje, Jugoslavia. (Communicated by C. Truesdell.)

1. Consider the differential equation

$$\frac{F''}{F} + (n^2 - 1)\frac{f''}{f} = 0, \qquad (n > 1) \tag{1}$$

where n is a positive integer, $F = F(z)$, $f = f(z)$, and primes denote differentiation with respect to z. P. Neményi and C. Truesdell[1] have reduced the general equilibrium problem in the membrane theory of shells of revolution to the integration of this single equation. I have recently given a procedure[2] which systematically yields cases in which (1) can be integrated by quadratures. In this note I present another method of integration for the equation (1).

2. By introducing the changes of variable

$$F = \exp\left(\int G dz\right), \qquad f = \exp\left(\int g dz\right),$$

we may put (1) into the form

$$(G - \lambda g)' + G^2 - \lambda g^2 = 0, \quad (\lambda = 1 - n^2),$$

or, equivalently,

$$\theta' + G^2 - \lambda g^2 = 0, \tag{2}$$

where we have put

$$G - \lambda g = \theta. \tag{3}$$

From equations (2) and (3) we have

$$G = \frac{-\theta \pm \Delta}{\lambda - 1}, \qquad g = \frac{-\lambda\theta \pm \Delta}{\lambda(\lambda - 1)}$$

$$(\Delta = + \sqrt{\lambda\theta^2 + (\lambda - \lambda^2)\theta'}),$$

where the sign $\pm$ is to be taken alike in the two expressions.

Consequently a solution of (1) is given by the formulae

$$F = A \exp\left(\int_{z_0}^{z} G dz\right), \qquad f = B \exp\left(\int_{z_0}^{z} g dz\right),$$

where A and B are two constants of integration, z_0 is a suitably chosen numerical constant. G and g are two functions of z defined by (4), in which there occurs a function of z, namely $\theta(z)$, which is completely arbitrary.

3. In a study now in press I apply the method of §2 to differential equations of a much more general type.

[1] Cf. NEMÉNYI, P., Bygningsstatiske Meddelelser, 1936; NEMÉNYI, P., and TRUESDELL, C., Proc. Nat. Acad. Sci. **29**: 159–162. 1943; TRUESDELL, C., Trans. Amer. Math. Soc. **58**: 96–166. 1945; **61**, 128–133. 1947.

[2] MITRINOVITCH, D. S., Comptes Rendus Acad. Sci. Paris **231**: 327–328. 1950.

ARCHEOLOGY.—*Notes on aboriginal pottery from Montana.*[1] WALDO R. WEDEL, U. S. National Museum.

Aboriginal pottery from Montana is no longer news to students of Plains prehistory and ethnography. Since 1940 its occurrence in various localities throughout the State has been noted several times in the archeological literature. Moreover, most historic Indian tribes of the region, including specifically the Blackfoot, Gros Ventres, Sarsi, Shoshoni, and Assiniboin, seem to have retained traditions recalling a time when they made pottery. Even the Crow, for whom ethnographers have apparently recorded no such traditions, are coming under progressively deeper suspicion of having been potters since their still undated arrival in the drainage of the Yellowstone.

The ethnographic data bearing on this subject have been well summarized, and some of their implications discussed, by Ewers (1945). The archeological evidences, which promise to give unsuspected historic depth to the native use of pottery on the headwaters of the Missouri River system, represent mainly materials and data gathered during the 1930's in course of work relief programs. Much of this latter information, which is of primary importance for the understanding of Montana prehistory, as well as some scattering data more recently gathered by River Basin Surveys parties, unfortunately still remains unpublished. From what is in print (Mulloy, 1942, 1945; Nelson, 1942, 1943), however, it is clear that pottery-using peoples have at one time or another scattered their traces, if thinly, throughout a number of Montana's stream valleys westward almost to the continental divide. Still to be determined, preferably on the basis of larger and better controlled samplings than are at present available, are the exact nature and chronological positions of the several pottery complexes apparently indicated, as well as the associated artifact types and subsistence economies. The location of the region in the heart of the erstwhile "potteryless" Plains culture area lends more than usual interest to the problems raised.

[1] Published by permission of the Secretary, Smithsonian Institution.

In the present discussion I do not intend to answer any of the questions implied above. My purpose is rather to put on record a few descriptive notes regarding several hitherto unreported or undescribed finds of pottery in northwestern Montana, all in the Missouri River watershed. These I have compared provisionally with materials already described from other localities lying mainly to the south and east, and also with what has been suggested or reconstructed from tribal traditions or historic documents regarding pottery of the historic Indians in the region. Though I have not personally examined the sites from which the material at hand is reported to have come, such information as has been furnished me seems to warrant the present notice.

The material immediately under consideration consists of small samples only and obviously does not give a complete picture of the material culture complex presumably represented in each case.[2] It includes a series of less than 100 sherds from a site near Ethridge, in Toole County; two smaller lots from two locations in Teton and Cascade Counties; and reports on two other sites in Chouteau and Cascade Counties. There is also a series of nearly 200 sherds from a cairn on the Crow Indian Reservation south of Billings. Only this latter series can be said to have been collected under anything like controlled conditions, or by a professional archeologist. The descriptive notes that follow are based on visual examination or, at most, on use of a hand lens.

[2] The Ethridge specimens were collected by Giles Ortscheid, formerly of Cut Bank, who turned them over to Claude Schaeffer, Museum of the Plains Indian, Browning, by whom they were forwarded to me. The specimens and records from Teton, Cascade, and Chouteau Counties were furnished by J. Robert Wells, formerly of Great Falls, to John C. Ewers, associate curator of ethnology, U. S. National Museum, who turned them over to me. The specimens from the Crow Indian Reservation were excavated by N. C. Nelson in 1941 for the American Museum of Natural History, and were sent me on loan by J. A. Ford, assistant curator of North American archeology at that institution. I am indebted to all these men, and particularly to Schaeffer, Ewers, and Ford, for their willingness to place these materials at my disposal and to supply relevant information.

Ethridge Site, Toole County.—In the sample forwarded to me from this site by Schaeffer there are approximately 100 sherds, plus arrowpoints, scrapers, other chipped flints, shell fragments, and, interestingly enough, three fragments of brass. They are from a camp site near a bison fall, situated 8 miles (airline) northwest of Ethridge, in the Marias River drainage. The bison bones and cultural materials are found at the lower end of a deep cleft in an east-facing escarpment some 200 feet high. According to Ortscheid (letter to Schaeffer, May 18, 1950), the sherds "seemed to be mostly 5 or 6 inches below the surface and some on the surface where the wind had blown clean spots. The copper or brass fragments were close to the surface. The stone chips and points . . . were mixed indiscriminately from surface to undisturbed soil, which varied from six inches to a couple of feet in places . . . most of the material was from the center of the campsite [which] covers several acres."

With regard to paste, color range, texture, and other technological details, examination with a hand lens seems to reveal no significant variation among the sherds at hand. Like much Northern Plains pottery, these usually have a gray to dark gray paste, occasionally fired to a light gray, brown, or buff on the exterior surface. Inclusions vary in amount and coarseness, even within individual sherds. Characteristically, they consist of crushed granite in medium to coarse angular particles, less commonly of rounded and waterworn gravel. They are usually only moderately abundant, and do not show on sherd surfaces. Exterior hardness varies from 3 to 4, occasionally reaching 4.5. Interior surfaces are rather rough and uneven. Carbonized material (presumed to be food remains) adhere to the inner surfaces of many fragments. The sherds vary in thickness from 6 to 12 mm; most are in the neighborhood of 8 mm. There are no recognizable base fragments, handles, or other constructional features, nor is there anything to indicate the range in vessel shapes and sizes.

With respect to surface finish and treatment, three main groups may be recognized. Fourteen sherds, including three rims from at least two vessels, are evidently fabric-marked (Fig. 1, A–F). These sherds range from 9 to 12 mm in thickness, which is considerably heavier than any other series present. The exterior surfaces have been strongly impressed by some fabric that left a dimpled texture, which occasionally somewhat suggests knotting or a net impression. Sherds in the national collections with strikingly similar surface treatment come from a Middle Woodland site near Gala, Va. (Holmes, 1903, pl. 133, upper right). This is an eastern Woodlands pottery trait; so far as my observations go, it is not common, and has apparently not been heretofore reported, in the Plains or on upper Missouri Valley potterywares.

Thirty-eight sherds, of which 16 are rim pieces, I have somewhat hesitantly classed as cord-roughened (Fig. 1, G–J). On some pieces there is no doubt that the impressions were made with a twisted cord or other fibrous element; in others, superficially very similar in appearance, the impressions do not conclusively show twisting of the element used. The group is rather variable; some sherds show deep fine impressions, whereas others have the impressions widely spaced or nearly obliterated. Despite this variability, most of the sherds would be assigned with little hesitation to the cord-roughened category, if they had been taken from a prehistoric Central Plains village site.

Eighteen sherds, all body fragments, are plain surfaced. Surfaces are moderately well smoothed, but often uneven and never polished. Occasionally there is some unevenness suggesting smoothed-over cord-roughening or possibly simple stamping.

Not included in the above counts, are about a dozen body sherds which can perhaps be described as weakly carinate. On each there is a perceptible shoulder bearing vertical or occasionally diagonal notches (Fig. 2, A–C), made with some thin-ended or triangulate-tipped instrument, and spaced usually at intervals of about ¼ inch. Most of these sherds are plain ware. Three are what I have tentatively called cord-roughened, and one of these latter bears diagonal notches made by impressing a cord-wrapped instrument (Fig. 2, G).

Significantly enough, there are in the whole series only two or three sherds, at most, which can be called simple stamped or fluted on the exterior surface. A number of small fragments cannot be satisfactorily identified as to surface treatment, and so are omitted from my counts and determinations.

Rims are characteristically thickened or somewhat bulbous in profile, with horizontal or outsloping flattened upper lip surface. Most of them seem to conform rather closely to Mulloy's rims

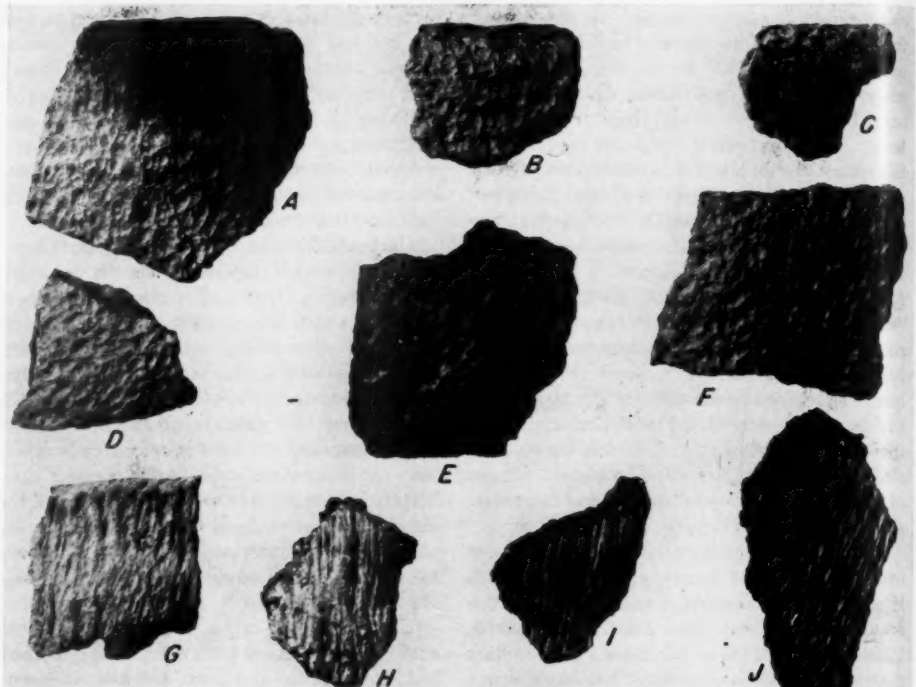

FIG. 1.—Potsherds from camp site near Ethridge, Mont.: A-F, fabric-impressed; G-J, cord-roughened.

of class A form at the Hagen site (Mulloy, 1942, p. 18). So far as I can see in the very limited sample at hand, there is no correlation between rim form, lip, or other related features and the type of surface treatment on the sherds. On cord-roughened rimsherds the flattened lip may be plain, weakly cord-roughened, or carelessly incised. On plain sherds the outer lip edge is sometimes notched; in two instances, the lip panel bears diagonal impressions made with a cord-wrapped rod (Fig. 2, D, F); and in a third the same treatment was applied with a loosely wrapped tool to the outer surface of the neck below the flat panel (Fig. 2, E).

The material culture complex of which the above described sherds were a part is inadequately represented by the specimens at hand. It seems to include, however, numerous small, triangular projectile points, averaging under 25 mm in length, with straight to slightly concave base and a single pair of side notches just above the base. Three or four are unnotched, and perhaps an equal number can be classed as stemmed or corner-notched; none shows base notching. Materials used include chalcedony, cherts of various colors, jasper, obsidian, and quartzite. All the points I have seen appear to be of types said to be common at many bison-kills of the Montana region, but with three exceptions they are perhaps somewhat less carefully made. They presumably represent the products of late prehistoric or protohistoric natives of the region.

There appears to be nothing distinctive about the scrapers and other chipped flints sent me from the Ethridge site.

The three pieces of brass, of course, must be attributed to Caucasian contact or influence, but it is not certain at the moment that they actually belong to the same complex as the sherds. The collector's letter (Ortscheid to Schaeffer, May 18, 1950) says they were "close to the surface" of the site. They are possibly intrusive; or they may

have been left by some later group than that responsible for the pottery. There is nothing to indicate the nature of the objects from which the fragments came.

Sites in Cascade, Teton, and Chouteau Counties.—Sherds from two sites in Cascade and Teton Counties, and notes on other occurrences in the same general region, were sent to Ewers in September 1947, by J. Robert Wells, then of Great Falls. Ten of the sherds are said to have been found in the summer of 1933 "near a bison trap on the south-facing sandstone escarpment of what is locally called the 'second bench' between Ulm and Vaughn, Cascade County." There is no further description of the locale of discovery.

The sherds are mostly small, the largest not exceeding two inches in maximum diameter. In thickness they range from 5 to 8 mm; in hardness, from 3 to 4. Paste again is gray to dark gray in color, with a fine granular appearance. The aplastic consists of gravel, mostly or entirely in rounded particles. The three largest sherds show impressions identical with what I have classed in the Ethridge material as cord-roughening on the exterior surface; and all but one of the remaining fragments, in color, texture, surface finish, and other particulars, seem to be from vessels of similar construction and appearance. The single exception, though small and inconclusive, shows several flutings somewhat reminiscent of nearly obliterated simple stamp impressions. Sherd interiors are usually uneven and only moderately well finished. There are no rim fragments.

From the second site, described only as being "near to Chouteau in Pondera [Teton] County . . . not over two miles approximately south from the town," there is a single large plain body sherd. This is remarkable chiefly for its thickness of 15 mm, which far exceeds any of the other Montana sherds which have come under my scrutiny. It has a gray core, which becomes light buff to brown on the surfaces. Temper consists of quartz, mostly in angular particles. The piece is well-fired and hard.

Wells's letter of June 11, 1945, to Ewers, first reporting these pottery finds, notes two other occurrences; from neither are there any specimens at hand. One of these finds was "in the 'cut bank' of the Missouri River at the Fair Grounds of Fort Benton, Chouteau County." Here, according to Mr. Wells, "I found remains suggestive of the interrupted preparation of a meal. Some 18 inches below the present land surface I noticed a short stratum of charcoal upon which rested what appeared to be the flattened fragments of a cooking

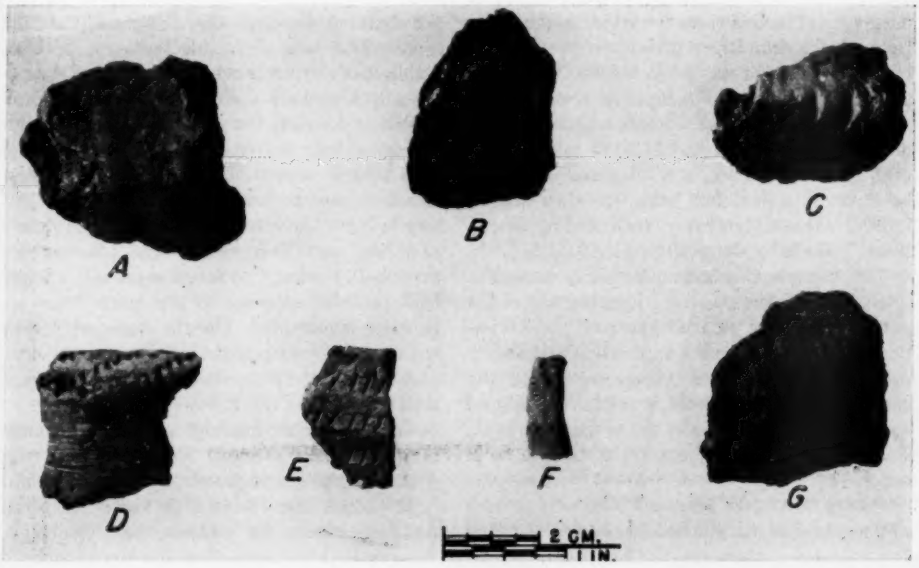

FIG. 2.—Potsherds from camp site near Ethridge, Montana. A-C, carinate with punctations; D-G, cord-wrapped rod-impressed.

pot mixed with several split bison bones. Preservation of the sherds was so poor that I am not sure they were kept. The presence of stone implements together with brass buttons upon about the same level might indicate a date near to the early historic period for these remains."

Also in Cascade County is another sherd-bearing site which "occupies the triangular piece of land west of the Missouri River and south of its tributary, the Sun River, near the mouth of the latter. This tract, now the grounds of the Meadowlark Country Club of Great Falls, was then (about 1920) under cultivation and a very fruitful source of artifacts especially in the fall and spring when the fields were bare. I had seen numerous sherds there on the surface before. . . . I recognized them for what they were. After this lapse of years I can only describe them as small, some ¼ to ⅜ inch in thickness, gray, fairly hard, and without sufficient shape to serve as criteria for judging as to the form of the vessels from which they came. If memory serves, the external surfaces, without exception, showed markings which I considered indicative of shaping with a paddle wound with cord approximately ¼ inch in diameter. . . . This site is now mostly occupied by the Country Club golf course, so it would seem unlikely that sherds can now be found there on the surface, but there is still a possibility of finding them embedded in the south bank of the Sun River which was then a 'cut bank' some five or six feet in height. That bank showed evidence of the occupancy of the site for what seemed to me an extended period (intermittent occupancy, I should have said). Charcoal, split bison bones, and lithic artifacts could be observed several feet below the then ground level. I do not, however, recall finding sherds there." (Wells to Ewers, June 11, 1945).

Two items in this last notice are of particular interest. One is the observer's identification of the pottery markings as cord-wrapped paddle impressions; the other is the implication that sherds occurred exclusively or preponderantly on the surface but were not noted in the buried cultural strata partially exposed in the nearby cut bank. Further investigations here would seem to be in order.

Pottery from the Yellowstone Valley.—My search of the published literature on Montana archeology has revealed pottery descriptions from four localities in the Yellowstone drainage basin. These include the Hagen site, 5 miles southeast of Glendive, in Dawson County, excavated by the Montana Archeological Survey and Works Projects Administration and since described in commendable detail and clarity by Mulloy (1942); a rock cairn at Arrow Rock on the Crow Indian Reservation some 35 miles south of Billings, excavated by Nelson (1942, 1943) for the American Museum of Natural History; Thirty Mile Mesa and Pompey's Pillar Cuesta, some 30 or 40 miles north and northeast of Billings; and Pictograph Cave, near Billings. I have seen none of the material from the Hagen site, but Mulloy's well-illustrated report is adequate for present purposes. As elsewhere noted, Nelson's Arrow Rock collection has been placed at my disposal. For the last three localities above enumerated, and findings there by the Montana Archeological Survey, brief notes by Mulloy (1945, p. 520) are available.

The Arrow Rock sherds were taken from a rock cairn situated on the floor of Pryor Canyon and partially trenched by Nelson. Here, in a "mixed earth and boulder deposit," it was found that "at least the upper half of the accumulation was moderately rich in the usual stone objects, glass beads and animal bones. More abundant were bone and shell ornamental items, chiefly beads and pendants. The surprise was the collection of some 200 potsherds." In the absence of word to the contrary, I assume that the glass beads and sherds came from the same levels or horizons within the cairn, and are to be regarded as chronologically associated finds.

Nelson describes the sherds as ranging in surface color from buff to gray, in body thickness from three to nine millimeters, and in rim thickness from nine to fourteen millimeters. The pottery is "sometimes tempered with coarse quartz particles," and "firing is well done." He suggests that "both bowl and jar forms, some with charred food particles adhering to the inside," are apparently represented. On the basis of surface treatment, he recognized three groups of sherds: plain surfaced, 128; corduroy surfaced, 67; and textile impressed (?), 2.

Compared to the Ethridge and Cascade County samples, Nelson's Arrow Rock sherds average generally smaller and thinner, few exceeding 6 mm in thickness. Many have little visible tempering material; others, by contrast, have quartz inclusions that may exceed in size the particles in the more northerly sherds. In surface color, the Arrow Rock sherds seem to run to somewhat

darker tones, seldom showing the light buff or tan exteriors found on the Ethridge sherds. Most of the fragments are appreciably more gritty to the touch than the Ethridge-Cascade County sherds. What Nelson calls "corduroy surfaced" sherds are, without question, Mulloy's fluted or what I have called simple stamped. The impressions, however, are much less regular and conspicuous than those produced by the same or a similar technique among the historic Mandan, Arikara, Pawnee, and other Plains potters. At least one, and quite possibly both, of Nelson's textile-surfaced sherds appear, from plasticene impressions, to be the same as the more plentiful fabric-impressed sherds from Ethridge.

The pottery from the Hagen site, according to Mulloy (1942, pp. 11–38), represents "a single rather well integrated cultural complex." It is described as having a granular, somewhat variable, paste; crushed rock or occasionally sand tempering; a hardness of 3 to 3.5; and a predominantly gray color. Medium-sized jars or ollas seem to be characteristic forms; rims are variable and include both "collared" and "uncollared" forms; lips are wavy or smooth, in the latter case sometimes bearing incised or impressed linear decoration. About half the sherds recovered were plain; another fifth bear fluted or simple stamped surfaces. Incised lines, wrapped-rod impressions, brush roughening, single-cord impressions, check stamping, and dentate stamping occur in decreasing order of frequency. In general technology, in vessel form, in design techniques (especially single-cord impressing, incising, and fluting), and in designs, the ware shows close similarities to the Mandan-Hidatsa pottery tradition. Wrapped rod impressions and dentate stamping, on the other hand, are not Mandan-Hidatsa, and suggest some other eastern influence, possibly on an earlier time level.

Elsewhere in the Yellowstone Valley, the pottery occurrences reported by Mulloy apparently involve, at least in part, wares whose relationships are with the Hagen site complex. Thus, he observes (Mulloy, 1945, p. 520) that "a few fragments of pottery were discovered near the house sites at both Thirty Mile Mesa and Pompey's Pillar Cuesta. They are gray to buff, with coarse paste and sand temper. In some the exterior is fluted, as though it might have been beaten with a thong-wrapped paddle. Pottery of this type occurs in small quantities on the surface in many places throughout this part of the Yellowstone Valley. It is similar to that of Pictograph Cave IV, an early historic occupation of Pictograph Cave, near Billings." Historic materials, including gun flints and trade beads, were also found at Thirty Mile Mesa; but since all these finds were apparently surface materials, exact associations remain obscure.

Comparisons.—The sherds from Ethridge, and those I have described from Cascade County, are closely similar to one another in all respects. Such variations as are apparent in paste, tempering, etc., do not appear to me to be of any great significance; possibly, if the Cascade County sample included as many fragments as are available from Ethridge, the similarity would be even closer. As it is, if the two samples were mixed, it would be impossible, I think, to separate them without recourse to identifying marks. Moreover, the techniques of surface treatment in these two lots are what I would consider predominantly prehistoric in character; a Plains archeologist, inspecting them without previous knowledge as to their provenience would, with little or no hesitation, at once suspect a late prehistoric horizon.

The Ethridge-Cascade County sherds, however, differ appreciably from Nelson's Arrow Rock material. The former are generally thicker, seem to be better fired and more carefully made, show a greater frequency of cord-roughening and fabric-impressing, and a much lower frequency or even near-absence of simple stamping. These differences, though not always easily verbalized, seem marked enough to set the two groups apart, even to a nonspecialist in Plains pottery types. To me, the Arrow Rock sherds, by contrast with the Ethridge-Cascade County materials, have a somewhat "decadent" look that is reminiscent of historically late Plains wares elsewhere—perhaps something like the differences between Dismal River and Upper Republican wares in the central Great Plains.

Having seen and handled none of the Hagen site pottery, I am at some disadvantage in attempting to compare it with the samples at hand. From Mulloy's published description, however, it would appear that with respect to paste, temper, and perhaps other technological details, no striking differences exist between Hagen site pottery and that from the Arrow Rock cairns or from the Ethridge-Cascade County sites. As concerns decorative treatment and surface finish, however, there are far fewer resemblances among the various series. In its heavy emphasis on simple

stamping, the Arrow Rock material is much closer to Hagen site than to the Ethridge-Cascade County sherds. The Hagen site sherds seemingly are thinner than those from Ethridge. Incised decoration, single-cord impressions, dentate-stamping, fluting, and check-stamping are very rare or absent from the Ethridge-Cascade County series; and with the exception of fluting (simple stamping), they are also absent from Nelson's Arrow Rock material. Wrapped-rod impressions occur, as Mulloy (1942, p. 37) has already noted, both at Hagen site and at Ethridge[3]; and so also, apparently, do punctates. More surprising, in light of present knowledge of Montana pottery, is the relative abundance of cord-roughening and fabric-impressing at Ethridge, both of which techniques are apparently absent from the much larger pottery series from the Hagen site. It will be interesting to see whether larger and more carefully controlled pottery samples from the Ethridge locality confirm the presence of these apparently distinguishing characteristics, and the possibly significant differences between pottery from the Yellowstone Valley and that from the Marias-Teton-Sun Rivers locality.

DISCUSSION

It is obvious, as I have already indicated, that the sherd samples under discussion here are an inadequate basis for any far-reaching conclusions or broad generalizations, although some speculation seems warranted. They do not suggest that the Montana region was ever one of intensive pottery-making; and it is perhaps significant that in the one stratified site reported to date as pottery-bearing, Pictograph Cave, sherds were found only in the uppermost deposits in association with white contact materials. If this suggested lateness and thinness of occupancy by pottery-making peoples is borne out by findings in other sections of the state, it will perhaps be possible in the not far distant future to allocate most of the ceramic remains to the immediate ancestors of one or another of the historic tribes of the area.

[3] Under date of May 7, 1950, Mulloy informs me as follows concerning his previous examination of a sherd series from Ethridge: "The sample was small and, as I recall, the design elements were typical of the Hagen site and done in cord-wrapped stick. Any of the sherds I saw could have been easily lost in a Hagen site sample. None had cord roughening or fabric impressions.' "

With regard to the materials reported from the Yellowstone Valley by Mulloy and Nelson, the former has argued cautiously but cogently for a possible Crow authorship. Recency is certainly strongly suggested by the heavy emphasis on such a relatively late ceramic trait as fluting (simple stamping), and by the occasional association of the sherds with trade beads. It is possible that some of the sherd occurrences, such as those noted by Mulloy at pole and log structures in the middle Yellowstone Valley, pertain to hunting parties from tribes normally residing farther to the east or northeast, as for example, the Hidatsa.

The Hagen site, of course, can not be so explained. It was obviously a village of some permanence and length of occupancy. The pottery complex has a great deal in common with that of the late pre-white contact Mandan and Hidatsa, including a number of nearly identical traits; and interesting parallels may be found in other aspects of the material culture complex of the two localities. Linguistic evidence and tribal traditions indicate that the Crow separated at a relatively late date from the Hidatsa; and it seems a not unreasonable view that the Hagen site perhaps represents one of their settlements on the move westward before they had sloughed off entirely their old semi-sedentary mode of village life and pottery-making and when they had not yet acquired the horse. If it can be shown ultimately that the Pryor Valley cairns, where pottery and glass beads occur together, are of Crow origin, the case for the Crow as pottery-using Indians will be considerably strengthened; but such evidence is apparently not yet at hand. At the moment, about all that can be said is that pottery decoratively treated in a style highly characteristic of historic Plains Indian wares is widely distributed throughout portions of Montana which have been, since at least the middle of the 18th century, the range of the Crow Indians; and since the Crow traditionally reached this area from one in which their closest linguistic relatives shared a well-developed pottery tradition, the Crow would seem to be an excellent prospect in the search for the native potters who left the late prehistoric and/or protohistoric ceramic remains in eastern and southern Montana.

But what of the sherd-bearing sites in the Marias-Teton-Sun Rivers area? Here there is no evidence of Crow penetration; and the sherds in the samples at hand seem to indicate something different from the late ware or wares to the south and southeast. What is suggested, moreover, is something quite unlike the crudely fashioned, flat-bottomed, subcylindrical pottery vessels reconstructed by Ewers (1945, p. 295) from historical accounts and tribal traditions of the Blackfoot, historic occupants of the region in question.[4] These latter products are somehow reminiscent of the early forms of metal vessels introduced by white traders, from which, indeed, they may even have been copied by Indians who probably knew nothing of the relatively better-made and technologically superior wares whose vestiges were to be found in some of the old campsites nearby. If the brass fragments at the Ethridge Site are actually associated with the sherds above described, it may be worth while to look further into the possibility of a Gros Ventre or Arapaho origin; but the feeling persists that we are perhaps dealing here with a tradition older than anything that might be associated with one or another of the historic tribes of northern Montana.

I have at the moment no suggestions to offer concerning the affiliations of the Ethridge-Cascade County sherds, except to say that to me they look eastern rather than western, prehistoric rather than historic. They are not strongly reminiscent of a rather mixed lot of sherds which I obtained for the Missouri River Basin Survey in 1947 at site 24RV1, on Big Muddy Creek southwest of Froid, in Roosevelt County, Montana, among which were noted thick, dentate-stamped sherds rather similar to certain Illinois Valley materials. Such traits as dentate-stamping, textile-impressing, cord-roughening, and the wrapped-rod technique seem to me to argue for a possible eastern Woodland cultural or ethnic thrust into the northwestern Plains, perhaps from the Minnesota region or elsewhere out of the western Great Lakes or upper Mississippi Valley area.

In summary, I am inclined to think that native pottery in the Montana region may be older than would be implied in the assumption that it was brought in by early peoples directly related to one or another of the known historic inhabitants of the region; or alternatively, that some of these arrivals from the east, such as the Arapaho, perhaps came at an earlier time than is commonly supposed. As Mulloy, in his discussion of the Hagen site and its implications, has remarked with reference to westward movements of peoples with a Mandan-Hidatsa culture tradition, although "the only westward movement of which we know is that of the Crow, it is entirely possible that a westward push such as the Crow movement may have taken place on several occasions in the prehistoric period. Small groups may have moved westward to live for a time and later to return or perhaps be absorbed by other groups." It is well established that farther south, in western Kansas, Nebraska, and eastern Colorado, prehistoric horticulturists and/or potters at one time or another pushed westward many hundreds of miles beyond the immediate valley of the Missouri and even into the High Plains proper. It seems possible that comparable thrusts westward by pottery-using Indians, perhaps during a Late Woodland culture period, may have taken place in Montana as well, though their stay in the region was evidently not as well marked as farther to the south. That full-scale horticultural economies accompanied pottery westward to the continental divide in Montana does not seem likely. The Hagen Site lies at or near the northwestern margin of lands climatically suitable for dependable maize gardening; the Marias-Teton-Sun Rivers locality is far beyond the area of known aboriginal maize-bean-squash horticulture.

Determination of the exact character of the subsistence pattern and general material culture complex of pottery-bearing sites in northern Montana would doubtless assist in

[4] In his letter to me, dated May 7, 1950, Mulloy notes the occurrence of another, possibly somehow related, flat-bottomed potteryware found in campsites in western Montana, in the Wyoming Basin, and in the Great Basin, sometimes in sites which also contain pottery in the Mandan-Hidatsa tradition. He suggests that this may have been brought into the Montana-Wyoming region by the Shoshoni. There is, so far as I know, no published report concerning this curious complex and its significance.

the orderly arranging of the later prehistoric records of human occupancy of the region. It is just possible, too, that here, as elsewhere in the Great Plains where systematic archeology has been done, current concepts of local prehistory would be shown to be in need of some overhauling.

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BOTANY.—*A new species of* Portulaca *from Okinawa.*[1]

EGBERT H. WALKER, U. S. National Museum, and SHINJUN TAWADA, Ryukyu Forestry Agency, Okinawa.

The junior author sent the annotated specimen, upon which this new species is based, along with miscellaneous collections from Okinawa Island, to the U. S. National Museum. He reports having first observed it 20 years ago at Zanpea-misaki (cape), Nakagami-gun, Okinawa Island, although the type is from near the village of Onna in Kunigami-gun, Okinawa Island. Search of the literature and comparison with available material indicates it probably represents an undescribed species. In view of the somewhat confused status of the recent literature on Japanese botany, the presentation of a new species from this area is somewhat hazardous.

According to the characterization of *Portulaca* in von Poellnitz's monograph,[2] this species seems to belong in the subgenus *Euportulaca* Speg., section *Rotundatae* von Poelln., subsection *Foveolatae* von Poelln. In habit, it resembles the widespread species *P. quadrifida* L., differing most significantly in the absence of axillary hairs. In this same respect, it differs from the two eastern Asiatic species, *P. insularis* Hosokawa[3] and *P. boninensis* Tuyama,[4] which have been published since von Poellnitz's monograph was issued. Its perennial caespitose habit, small leaves, absence of axial hairs, so characteristic of many species, and its foveolate sculptured seeds are the outstanding characteristics of this species.

Portulaca okinawensis Walker & Tawada, sp. nov.

Planta perennis caespitosa 5–10 cm alta, caulibus herbaceis numerosis viridibus implicatis ramosis e caule brevi lignoso griseo orientibus; radicibus non visis; foliis ramulorum apices versus plerumque enatis, subsessilibus vel petiolatis, alternatis vel sub flore vel fructu solitario terminali verticillatis; laminis foliorum in vivo crassis carnosis in sicco dense granulosis elliptico-ovatis vel oblongis, 2–4 mm longis, basi obtusis, apice obtusis vel rotundatis, margine integris; foliis involucralibus non carinatis; pilis axillaribus nullis; floribus solitariis terminalibus circiter 1.6 mm diametro, petalis 6, aurantio-flavis vel rubescentibus, staminibus circiter 25, liberis, pistillo solitario vix inferiore, stylo gracili superne paullo dilatato, stigmate 4-partito; fructu globoso 2–3 mm diametro nitido horizontaliter dehiscente, cupulae basalis seminiferae margine plus minusve incrassato, semine minuto atro nitido foveolis numerosis non profundis facie ornato.

Nom Jap. Okinawa-matsube-botan (ex Tawada).

Type in the U. S. National Herbarium, no. 1992668, collected October 2, 1949, by Shinjun Tawada (no. 2221) on an exposed rock at the seaside, 20 feet elevation, at Onna, Kunigami-gun, Okinawa Island, in the Ryukyu Islands.

[1] Published by permission of the Secretary of the Smithsonian Institution.

[2] POELLNITZ, K. VON. *Versuch einer Monographie der Gattung Portulaca L.* Repert. Sp. Nov. Fedde **37**: 240–320. 1934.

[3] Trans. Nat. Hist. Soc. Formosa **22**: 229. 1934.

[4] Bot. Mag. Tokyo **53**: 6. 1939.

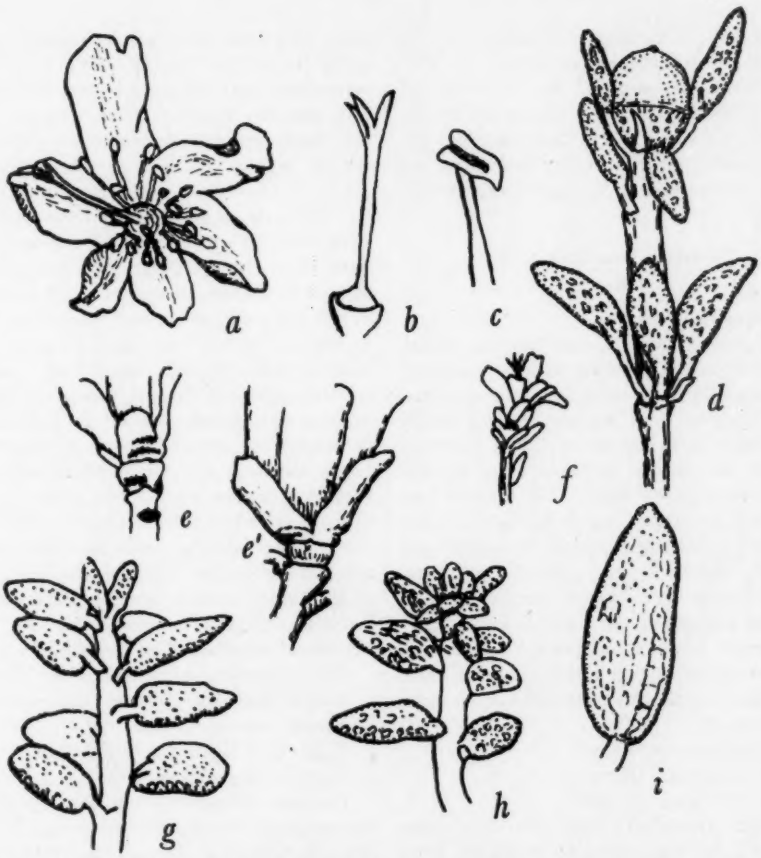


FIG. 1.—*Portulaca okinawensis* Walker & Tawada, sp. nov.: *a*, Flower; *b*, pistil; *c*, stamen; *d*, fruiting stem tip; *e*, *e'*, flowering stem tip showing reddish bases of bracts; *f*, flowering stem; *g*, vegetative stem, lower side; *h*, vegetative stem, upper side; *i*, enlarged leaf, showing greenish intermittent net forming margins to translucent center. Drawing by Tawada.

ENTOMOLOGY.—*New species of Gelechiidae from Argentina* (*Lepidoptera*). J. F. GATES CLARKE, Bureau of Entomology and Plant Quarantine.

The following species of Gelechiidae are described from material submitted by Fernando Bourquin and J. A. Pastrana, of Buenos Aires. Two species were reared by Mr. Bourquin and one by Mr. Pastrana from larvae they collected. Figures of the moths and life history notes will be published by Mr. Bourquin.

Parastega hemisigna, n. sp.
Fig. 1

Alar expanse, 16 mm.

Labial palpus, antenna, head, thorax, tegula, and ground color of forewing dark, shining purplish-fuscous. Brush of second segment gray and extreme apex of third segment creamy white. From costa of forewing, at one-fifth, a white bar extends to fold, then is continued along the fold to tornus as a narrow tawny line; extreme base of wing and an elongate patch beyond the white bar, black; costal edge, beyond white bar, and cilia, sooty. Hind wing gray; cilia fuscous. Forelegs and midlegs dark purplish fuscous with narrow white annulations on tarsi; hind leg sooty with narrow white annulations on tibia and tarsus. Abdomen shining blackish fuscous; anal tuft sordid ocherous-white with dull fuscous scales mixed ventrally.

Male genitalia.—As figured.

Female genitalia.—Unknown.

Type.—U.S.N.M. no. 60941.

Remarks.—Described from the type male dated, "VI. 50" and reared by Fernando Bourquin.

Similar in size to the Central American *P. chionostigma* (Walsingham) and *P. niveisignella* (Zeller) but distinguished from the former by the dark head and palpus and from the latter by the absence of the brownish scaling of forewing.

I have figured (Figs. 2, 2a) the uncus, gnathos, and right harpe of the type of the genus (*niveisignella*) for comparison.

Mr. Bourquin has two additional specimens of *hemisigna* and writes that "the male has two white stripes and the female one white stripe."

Aristotelia perplexa, n. sp.
Figs. 3-3a, 4

Alar expanse, 10–12 mm.

Labial palpus whitish, pink tinged; second segment with brownish-ocherous median and subapical bands; third segment with broad fuscous submedian and subapical bands. Antenna fuscous narrowly banded with white except dorsally the bands not forming complete rings. Head, thorax, tegula, and base of forewing brownish ocherous. Ground color of forewing sordid whitish, the scales tipped with fuscous; basal patch broadly edged with dark brown outwardly; from basal third of costa a dark brown oblique band extends to slightly beyond fold and beyond this, in cell, is a small fuscous spot followed by another at the end of cell; outer half of wing overlaid with brownish ocherous; apical half of costa and termen edged with fuscous, the line broken by a series of pale carmine spots; cilia light brownish ocherous with subterminal and subbasal fuscous bands and base pale carmine; underside fuscous. Hindwing fuscous; cilia slightly paler; from costa of male extends a thick brownish-ocherous hair-pencil. Legs shining ocherous-white variously overlaid and banded with fuscous; foretibia and midtibia and tarsi and posterior tibia alternately banded with pale carmine. Abdomen fuscous above and ocherous-white beneath.

Male genitalia.—As figured.

Female genitalia.—Genital plate and ostium as figured; signum absent.

Type.—U.S.N.M. no. 60942.

Type locality.—Tigre, Argentina.

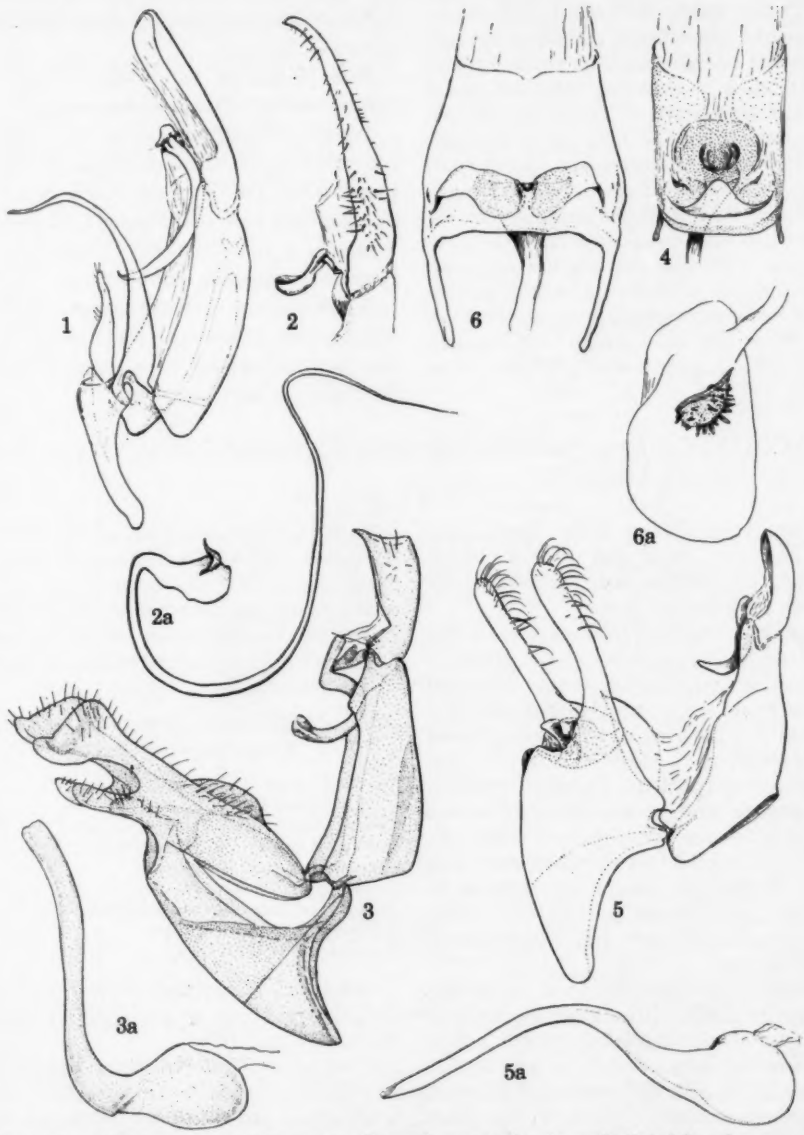
Remarks.—Described from the type male and five male and two female paratypes, all from the same locality. The dates on the type series are from March to April 1939. Paratypes in U. S. National Museum and Mr. Bourquin's collection, Buenos Aires.

This species is similar to *A. cynthia* Meyrick and possesses the hair-pencil from costa of hind wing of male; but *cynthia* lacks the carmine coloring of *perplexa*. The cucullus of *cynthia* is greatly elongated and sharply curved ventrad, while that of *perplexa* is short and dilated.

Aristotelia parephoria, n. sp.
Figs. 5-5a, 6-6a

Alar expanse, 11–14 mm.

Labial palpus sordid white; second segment with three bands and apex light brown; third segment with basal and median bands light brown and subapical annulation blackish fuscous. Head



FIGS. 1–6a.—1, *Parastega hemisigna*, n. sp.: Lateral aspect of male genitalia with aedeagus removed. 2–2a, *Parastega niveisignella*(Zeller): 2, Lateral aspect of uncus and gnathos; 2a, right harpe. 3–3a, *Aristotelia perplexa*, n. sp.: 3, Lateral aspect of male genitalia with aedeagus removed; 3a, aedeagus. 4, *Arsitotelia perplexa*, n. sp.: Detail of genital plate and ostium. 5–5a, *Aristotelia parephoria*, n. sp.: 5, Lateral aspect of male genitalia with aedeagus removed; 5a, aedeagus. 6–6a, *Aristotelia parephoria*, n. sp.: 6, Detail of genital plate and ostium; 6a, bursa copulatrix and signum.

pale brownish ocherous with a dorsal fuscous stripe. Thorax, tegula, and ground color of forewing ocherous-white; thorax and tegula strongly suffused with fuscous anteriorly; dorsal half and apex of forewing overlaid with buff; from base of costa, and from costa at one-third, blackish-fuscous bands extend to fold, the latter band, outwardly curved, joins narrowly a fuscous shade at outer third of costa; apex and tornus each with a small fuscous shade extended into the otherwise buff cilia; underside of forewing blackish fuscous. Hind wing and cilia fuscous; costal third of underside of hind wing blackish fuscous, remainder ocherous-white. Legs shining ocherous-white; tibiae and tarsi banded with blackish fuscous, abdomen grayish above, ocherous-white beneath.

Male genitalia.—As figured.

Female genitalia.—Genital plate, ostium, and signum as figured.

Type.—U.S.N.M. no. 60943.

Type locality.—Tucumán, Argentina.

Remarks.—Described from the type male and two male and four female paratypes, all from the type locality. The dates are "VIII, 1939," and the specimens were collected by J. A. Pastrana. Paratypes in U. S. National Museum and Mr. Pastrana's collection, Buenos Aires.

A. parephoria appears to be nearest to *A. ephoria* Meyrick but differs from that species by the long terminal segment of palpus and the three dark bands on second segment.

MALACOLOGY.—*Recent species of the cyrenoid pelecypod* Glossus.[1] DAVID NICOL, U. S. National Museum.

The study on *Glossus* is the fourth of a series on relict pelecypod genera. Lamy (1920, pp. 290–296) has done the most recent thorough work on the genus.

Glossus is represented by one species living in western European seas and the Mediterranean. The Indo-Pacific species allocated to *Meiocardia* have entirely different geographical distributions and certainly should be considered as a distinct genus on the basis of shell morphology. The exact relationship between *Glossus* and *Meiocardia* has never been shown, although Dall (1900, pp. 1065, 1066) claimed that the fossil and living species of the two groups are difficult to separate. Dall, Bartsch, and Rehder (1938, p. 121) consider *Glossus* and *Meiocardia* as distinct genera.

The torsion of the beaks has so greatly modified the hinge of the glossids that it is difficult to allocate the family to any higher taxonomic category, and it is not certain that any of the Mesozoic species of glossoid-form pelecypods can be placed in the genus *Glossus*. (See Stoliczka, 1871, p. 188.) Despite the great amount of torsion in *Glossus*, however, the genus bears much superficial resemblance to *Arctica*. This resemblance would be even more striking if the hinge of *Arctica* were twisted to the same degree that it is in *Glossus*.

From the Paleocene through the Miocene, the genus *Glossus* has apparently been confined to temperate seas in the northern hemisphere except for the northern Pacific region. From the Pliocene to the Recent, the genus has been confined to western Europe and the Mediterranean Sea.

Family GLOSSIDAE Stoliczka, 1871

Genus **Glossus** Poli, 1795

Cardium Linné, 1758 (in part).
Chama Linné, 1764 (in part).
Chama Linné, 1767 (in part).
Cardita Bruguière, 1792 (in part).
Glossoderma Poli, 1795.
Isocardia Lamarck, 1799.
Buccardium Megerle von Mühlfeld, 1811.
Bucardia Schumacher, 1817.
Tychocardia Römer, 1869.

Genotype: (Monotypy) *Glossus rubicundus* Poli, 1795 = *Chama cor* Linné, 1767 = *Cardium humanum* Linné, 1758.

There appears to be no nomenclatorial reason why Poli's names can not be used despite the fact that he employed two generic names, one for the soft parts of the mollusk and the other for the shell. The shell name always ends in "derma," and *Cerastoderma* has been used consistently for a genus of cardiids. *Glossus* and *Glossoderma* are absolute synonyms, but *Glossus* is to be preferred on the basis of page priority. *Glossus* Poli, 1795, is clearly prior to *Isocardia* Lamarck, 1799, and on that basis must be employed for *Cardium humanum* Linné.

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Glossus humanus (Linné), 1758

Figs. 2–5

1758 *Cardium humanum* Linné, Syst. Nat., ed. 10: 682.

1764. *Chama cordiformis* Linné, Mus. Lud. Ul. Reg.: 516.

1767. *Chama cor* Linné, Syst. Nat., ed. 12, **1** (pt. 2): 1137.

1792. *Cardita cor* (Linné), Bruguière, Encyc. Meth., Nat. Hist. Vers, **1**: 403, 404; 1797, *Cardita*, pt. 19, no. 18: pl. 232, figs. 1a–d.

1795. *Glossus rubicundus* Poli, Test. utr. Siciliae **2**: 114, 253, pl. 15, figs. 30, 34, 35, 36; pl. 23, figs. 1, 2.

1795. *Glossoderma rubicundus* Poli, Test. utr. Siciliae **2**: 253.

1795. *Glossoderma cor* (Linné), Poli, Test. utr. Siciliae **2**: 259.

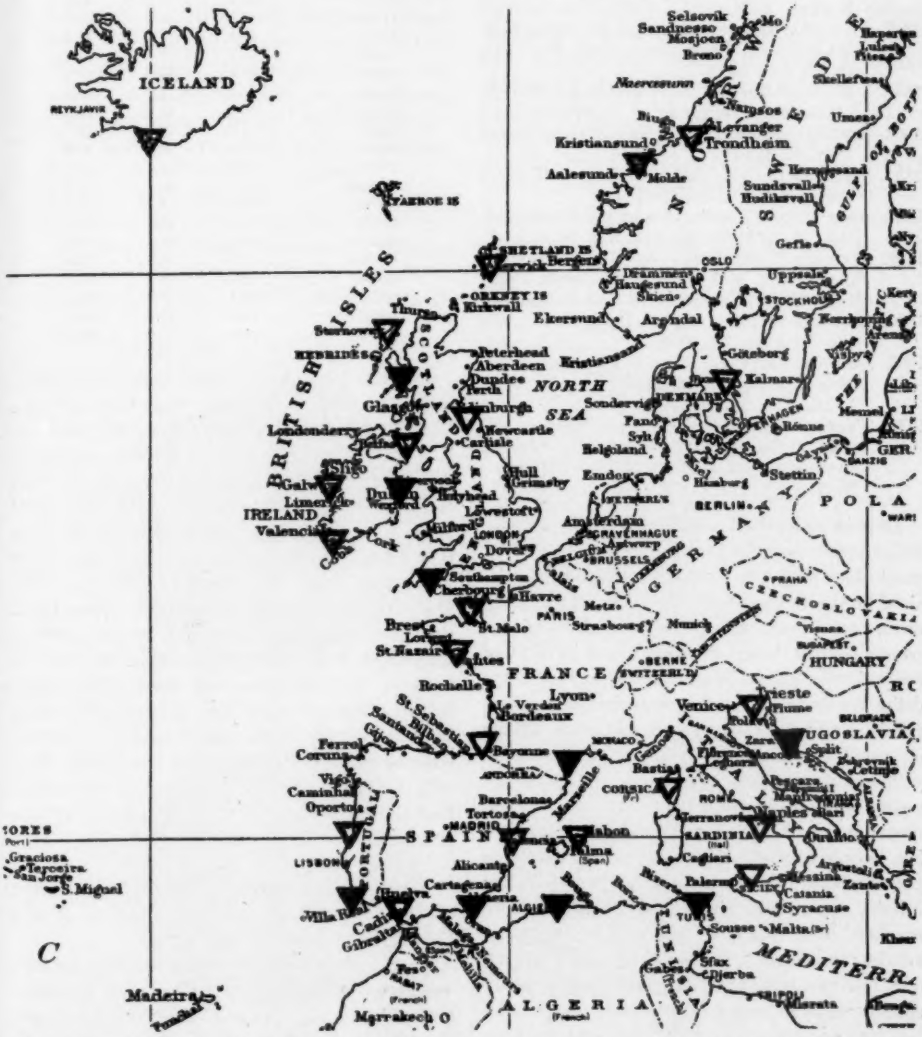


FIG. 1.—Distribution of living specimens of *Glossus humanus* (Linné): ▼, Locality data based on U. S. National Museum specimens; ▽, locality data based on specimens in other museums and on published records.

1798. *Cardium cor auritum* Röding, Mus. Bolt.: 192, no. 415.
1799. *Isocardia cor* (Linné), Lamarck, Mem. Soc. Hist. Nat. Paris **1**: 86.
1801. *Isocardia globosa* Lamarck, Syst. animaux sans vert., etc., **1**: 118.
1811. *Buccardium commune* Megerle von Mühlfeld, Mag. Ges. Nat. Freunde Berlin **5** (1), art. 2: 52.
1815. *Glossus cor* (Linné), Oken, Lehr. Nat., Teil 3, Zool.: 235.
1817. *Bucardia communis* (Megerle von Mühlfeld), Schumacher, Essai nouv. syst., etc.: 144, pl. 13, figs. 2a, b.
1845. *Isocardia hibernica* Reeve, Conch. Icon. **2**, *Isocardia:* pl. 1, sp. 4.
1853. *Cardita humana* (Linné), Mörch, Cat. Conch. Yoldi **2**: 38.
1855. *Isocardia cor* var. *hibernica* Reeve, Hanley, Ipsa Linnaei Conchylia: 84.
1858. *Bucardia cor* (Linné), H. and A. Adams, Gen. Rec. Moll. **2**: 461, pl. 112, figs. 5a, b.
1869. *Isocardia* (*Tychocardia*) *cor* (Linné), Römer, *in* Martini und Chemnitz, Conch.-Cab., ed. 2, **10** (2), Cardiacea: 5–7, pl. 1, figs. 1–3.
1900. *Isocardia humana* (Linné), Dall, Tert. fauna Florida **3** (pt. 5): 1064.
1903. *Isocardia cor* var. *valentiana* Pallary, Ann. Mus. Hist. Nat. Marseille, Zool., **8**, mém. 1: 15, pl. 1, fig. 13.
1937. *Glossus humanus* (Linné), van Regteren Altena, Bijdrage tot de kennis der fossiele, subfossiele en Recente Mollusken, etc.: 70, 71.

Description.—Shell porcellaneous, thin, exterior ornamented only by growth lines; small specimens often have two small folds separated by incised lines extending from the ligament obliquely downward toward the posteroventral margin; a poorly defined, broad depressed area in front of beaks, better defined in small specimens and often delimited by two incised lines; periostracum dark reddish brown to black on large specimens, lighter on small specimens, attaining a light greenish yellow on smallest specimens; periostracum nearly smooth in appearance on large specimens; on small ones fine, closely spaced, radiating lines composed of darker-colored ridges of periostracum; valves without gape, equivalve; interior ventral margin smooth; beaks spirally enrolled and strongly prosogyrate, umbones swollen; ligament external, weak, parivincular, split into two parts anteriorly and dragged under spirally enrolled beaks, opisthodetic; hinge teeth cyrenoid, hinge formula $\frac{3a, 1, 3b, PI}{2a, 2b, 4b, PII}$, all teeth laminar and nearly horizontal, 2a and 2b in left valve almost completely fused, 1 and 3b in right valve somewhat fused; pallial line integripalliate, anterior adductor muscle scar deeper and better marked but smaller than posterior adductor muscle scar.

Measurements in mm.—Only specimens with both valves were measured:

U.S.N.M. no.	*Length*	*Height*	*Convexity (both values)*
201293	104.2	98.3	83.2
201292	100.8	95.4	88.4
131658	93.0	80.2	77.7
304722	83.7	74.7	65.5
201294	81.7	79.6	59.0
201290	81.2	80.0	60.0
186122	79.3	74.7	69.5
201295	75.1	75.6	59.1
201302	70.3	64.1	59.7
304782	69.6	65.1	59.0
201298	67.8	66.1	61.3
201298a	60.4	59.6	53.2
201296	52.6	55.8	40.3
201300	45.9	48.8	36.2
201294a	45.6	48.2	35.7
131658a	45.6	45.6	38.6
201296a	44.1	48.0	34.4
201297	38.9	43.0	29.0
201297a	32.8	35.0	24.3
201337	26.8	27.7	20.5

One trend is quite apparent from the measurements: small shells are longer than they are high, whereas large shells are higher than they are long. The ratio of convexity to height was computed. All seven shells from the Mediterranean Sea had ratios ranging from 0.90 to 0.84. The ratios of 12 shells from the British Isles ranged from 0.80 to 0.72 except for one large shell from Dublin Bay which had a ratio of convexity to height of 0.88. Reeve (1845, vol. 2, p. 2, *Isocardia*) claimed that the specimens from Ireland were less globose than those from the Mediterranean Sea. On the basis of this difference and some other minor features, he proposed the new species name *hibernica* for the Irish specimens. To my knowledge no other conchologist has considered *hibernica* a distinct species, but Reeve's contention that the Mediterranean specimens are more globose is borne out by the few specimens I have measured.

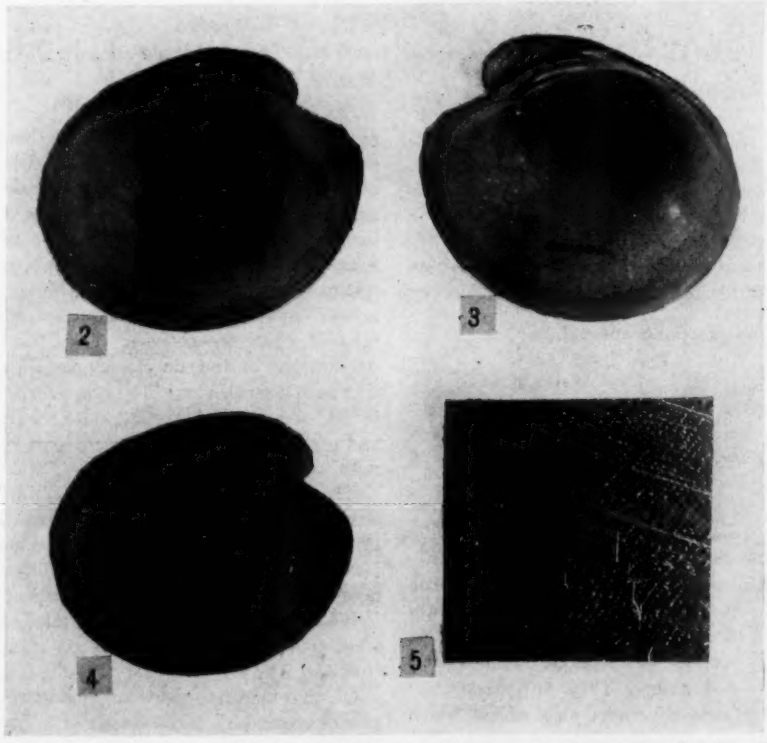
Number of specimens.—There are 32 specimens of *Glossus humanus* in the collection of the United States National Museum.

Locality data.—The following localities are represented by specimens in the National Museum: Zara, Yugoslavia; Tunis; Cette, France; Algiers; Cape de Gata, Spain; Cape Sagres, Portugal; Falmouth, England; Plymouth, England; Dublin Bay; Isle of Man; Oban, Scotland; Hebrides; Shetland Islands.

GEOGRAPHICAL DISTRIBUTION AND ECOLOGY OF GLOSSUS HUMANUS (LINNÉ)

This study is encumbered by two difficulties. *Glossus humanus* is not a common species, except for a few scattered localities, and observations on its habitat are meager. The more serious difficulty results from an error by J. Gwyn Jeffrys, who mistook species of *Kelliella* for the young of *Glossus*. Some of the Jeffreys' material collected on the *Porcupine* and *Valorous* expeditions is in the National Museum collection. Specimens identified as "*Isocardia cor*" by Jeffreys are not that species, a point upheld by Sars and much later by other conchologists.

The exact northern limit of distribution of *Glossus* is worthy of much additional investigation. Only one living specimen of *Glossus humanus* has been found off the southern coast of Iceland thus far (Madsen, 1949, p. 49), although the molluscan fauna of the island has been extensively collected and studied. The genus has not been reported from the Faroes. The report of *Glossus* from the Lofoten Islands off the coast of Norway was based on a misidentification by Jeffreys. The genus is rare from Trondhjem Fjord southward and eastward into the Kattegat. *Glossus* is fairly common in certain places along the coasts of the British Isles and is also found in the Shetlands. It has been reported all along the coasts of France, Portugal, and Spain. In the Mediterranean, *Glossus* is frequently found as far east as the Adriatic Sea. The fact that it has not been found east of there may be due to lack of careful collecting. It apparently is not present on the west coast of Africa, even near the entrance to the Mediterranean Sea. Jeffreys has reported *Glossus* from the Azores, but this report is



FIGS. 2–5.—*Glossus humanus* (Linné): 2, Interior of left valve, × 1; 3, interior of right valve, × 1; 4, exterior of right valve, × 1; 5, enlarged portion of exterior surface of shell showing fine radial ridges of periostracum, × 6. (All figures are of a young specimen from Falmouth, England; U.S.N.M. no. 201300.)

thought to be based on a misidentification. Further collecting will no doubt more accurately delimit the distribution of the genus.

Additional ecological data are greatly needed on *Glossus humanus*. The species apparently is found on sand, sandy-mud, or mud bottoms. It has been thought by some to have a wide bathymetric range, but this idea is now believed to be incorrect. Jeffreys has reported *Glossus* from more than 2,000 meters of water, but the specimens found at that depth are probably all *Kelliella*. *Glossus* apparently is found in depths ranging from about 5 to 150 meters. The probable temperature of the bottom where the genus thrives ranges from 8° to 15°C.

Acknowledgments.—The following persons gave me data on geographical distribution of specimens of *Glossus*: William J. Clench, Museum of Comparative Zoology at Harvard College; Leo G. Hertlein, California Academy of Sciences; A. Myra Keen, Stanford University. I am greatly indebted to them for their assistance.

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MALACOLOGY.—*More new urocoptid mollusks from Mexico.* PAUL BARTSCH, U. S. National Museum.

To the indefatigable efforts and the stimulating influence that Miss Marie Bourgeois, of Mixcoac, exerted upon her friends to help make known the molluscan fauna of Mexico, the U. S. National Museum is indebted for the following new species of urocoptid land snails transmitted to us for report.

Coelostemma anconai, n. sp.
Figs. 1, 3

Shell cylindroconic, pale horn-colored when living, dead shells white. The nucleus consists of about two turns, which are somewhat inflated and strongly rounded and form a slightly bulbous apex. The nuclear turns are finely granulose. The first seven postnuclear whorls increase gradually in width, rendering this part of the shell elongate-conic. Beginning with the eighth turn the shell becomes cylindric in form, contracting slightly on the last three whorls. The postnuclear whorls are slightly rounded and separated by a moderately impressed suture. They are marked by decidedly retractively curved axial riblets, which are slightly less strongly developed on the cylindric portion of the shell than on the two ends. Of these riblets about 40 are present on the second postnuclear turn, 80 on the tenth, and 62 on the penultimate whorl. On the last turn behind the peristome the riblets become fine, hairlike, and crowded. The spaces separating the riblets average about double the width of the ribs. The last turn is solute for about one-fifth of a turn, the solute portion bearing the rib sculpture of the rest of this portion of the shell. The aperture is subcircular and is somewhat sinuous on the parietal wall where the peristome is a little less expanded than on the rest of the aperture where it widens in a gentle curve. The columella is hollow, broad, about one-third the width of the shell, and shows fine axial markings; it gradually narrows in the last two turns.

The holotype, U.S.N.M. no. 595018, has 19 whorls and measures: Length 26 mm; diameter of the cylindric portion 6 mm. U.S.N.M. no. 595019 comprises the paratype, of which we have figured the columella and some fragments.

We are naming the species for Prof. I. Ancona, who collected the specimens at Ixcatiopan, Guerrero, Mexico.

Of the known species of *Coelostemma* this species resembles most nearly *C. igualaensis* Bartsch, from Iguala, Guerrera, Mexico, from which it is easily distinguished by its smaller size, more cylindric outline, narrower shell, and stronger ribbing.

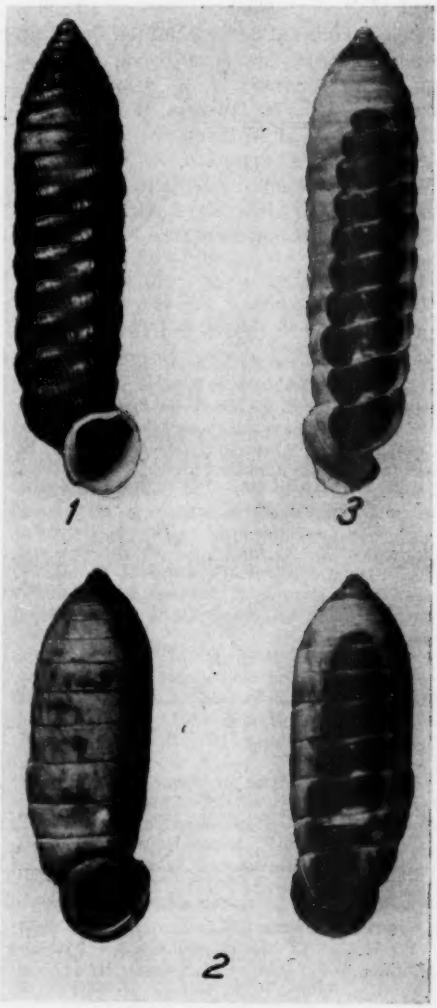
Holospira wilmoti, n. sp.
Fig. 2

Shell cylindroconic, white with the interior of the aperture pale chestnut-brown. The nucleus consists of about 2¼ strongly rounded whorls that form a mucronate apex. The first four postnuclear whorls increase rapidly in width, while the succeeding turns are cylindric, contracting again toward the base. The postnuclear whorls are flattened and separated by a slightly impressed suture. On the conic portion feeble decidedly retractively curved axial riblets are indicated, while on the cylindric portion the axial markings are reduced to mere lines of growth. The last whorl and a little of the penultimate turn bear distantly spaced somewhat sinuous axial ribs, which extend undiminished over the slightly angulated periphery and the base into the umbilical chink. These ribs are about one-third as wide as the spaces that separate them. The last whorl is solute for about one-eighth of a turn. Aperture obliquely pear-shaped; peristome broadly flatly expanded and thickened. Columella hollow, about one-fourth the diameter of the whorls, bearing a feeble obsolete fold in the cylindric portion of the shell which expands into a thin slightly curved blade in the penultimate whorl, where it extends over three-fifths of the width of the chamber bending slightly upward toward the parietal fold. In the last turn the columellar fold becomes much reduced and thickened, being scarcely noticeable in the aperture. The parietal fold is well developed and is confined to the penultimate turn. The basal fold in the same turn is poorly developed, while the labial fold is about one-half as strong as the parietal fold.

The type, U.S.N.M. no. 595020, was collected by George Wilmot on Cerro del Fraile, near Villa García, Nuevo León, Mexico. It has 14 whorls and measures: Length 20 mm; diameter of cylindric portion 7 mm.

This species most nearly resembles *H. orcutti* Bartsch, which Orcutt collected on a limestone paredon in Coahuila, Mexico. Its much smaller size and more cylindric form readily distinguish it.

We take pleasure in naming it for its discoverer.



FIGS. 1–3.—1, 3, *Coelostemma anconai*, n. sp.; 2, *Holospira wilmoti*, n. sp.

PROCEEDINGS OF THE ACADEMY

444TH MEETING OF BOARD OF MANAGERS

The 444th meeting of the Board of Managers, held in the Cosmos Club on February 12, 1951, was called to order at 8:05 P.M. by the President, NATHAN R. SMITH. Also present were: W. RAMBERG, H. S. RAPPLEYE, J. A. STEVENSON, C. DRECHSLER, A. T. MCPHERSON, W. R. WEDEL, J. J. FAHEY, E. H. WALKER, W. A. DAYTON, R. S. DILL, L. A. SPINDLER, A. M. GRIFFIN, F. M. DEFANDORF, and, by invitation, MARGARET PITTMAN, G. P. WALTON, and L. E. YOCUM.

The President announced the following appointments:

Appointed Members of Executive Committee: W. Ramberg, H. S. Rappleye, J. A. Stevenson, and F. M. Defandorf.

Board of Editors of the Journal: Charles Drechsler, to replace Frederick J. Hermann, designated Senior Editor. J. P. E. Morrison to replace F. C. Kracek. T. P. Thayer (Geology) was appointed an Associate Editor for a term of two years; R. K. Cook (physics and mathematics), F. A. Chace (biology), and M. L. Bomhard (botany) were appointed Associate Editors for a term of 3 years.

Committee on Membership: L. A. Spindler (Chairman), M. S. Anderson, Merrill Bernard, R. E. Blackwelder, R. C. Duncan, George T. Faust, Ira B. Hansen, D. Breese Jones, Dorothy Nickerson, Francis A. Smith, Heinz Specht, Alfred Weissler.

Committee on Meetings: Margaret Pittman (Chairman), Norman Bekkedahl, W. R. Chapline, Dorland J. Davis, F. B. Scheetz, Henry W. Wells.

Committee on Monographs: J. R. Swallen (Chairman). To January 1954: S. F. Blake, F. C. Kracek.

Committee on Awards for Scientific Achievement: George P. Walton, General Chairman. *For the Biological Sciences:* G. H. Coons (Chairman), J. E. Faber, Jr., Myrna F. Jones, F. W. Poos, J. R. Swallen. *For the Engineering Sciences:* R. S. Dill (Chairman), Arsham Amirikian, J. W. McBurney, Frank Neumann, A. H. Scott. *For the Physical Sciences:* G. P. Walton (Chairman), F. S. Brackett, G. E. Holm, C. J. Humphreys, J. Howard McMillen. *For Teaching of Science:* B. D. Van Evera (Chairman), R. P. Barnes, F. E. Fox, T. Koppanyi, M. H. Martin, A. T. McPherson.

Committee on Grants-in-Aid for Research: L. E. Yocum (Chairman). M. X. Sullivan, H. L. Whittemore.

Committee on Policy and Planning: J. I. Hoffman (Chairman). To January 1954: Henry B. Collins, Jr., W. W. Rubey.

Committee on Encouragement of Science Talent: M. A. Mason (Chairman). To January 1954: J. M. Caldwell, Waldo L. Schmitt.

Representative on Council of A.A.A.S.: F. M. Setzler.

Committee of Auditors: J. H. Martin (Chairman), N. F. Braaten, W. J. Youden.

Committee of Tellers: W. G. Brombacher (Chairman), A. R. Merz, Louise M. Russell.

The Secretary reported a meeting of the Executive Committee at 6:30 P.M., February 12, 1951, at the Cosmos Club with the following members in attendance: N. R. SMITH, W. RAMBERG, H. S. RAPPLEYE, J. A. STEVENSON, and F. M. DEFANDORF. At this meeting the budget presented by the Treasurer for 1951 was discussed in some detail and approved for submission to the Board of Managers.

The following budget was presented to the Board, discussed by the Treasurer, and adopted by the Board without change:

RECEIPTS

	1950	*Estimated 1951*	*1951 Budget*
Dues	$3985.00	$4200.00	
Journal Subscriptions	1339.50	1600.00	
Interest & Dividends	1521.50	1950.00	
Sales	93.43	100.00	
	$6939.43	$7850.00	

DISBURSEMENTS

Journal & Journal Office	$6435.74	$6500.00	$6500.00
Secretary's office	481.26	550.00	550.00
Treasurer's office	304.59	300.00	300.00
S. M. & C. of Publications	30.67	50.00	50.00
Meetings Committee	244.30	500.00	500.00
Membership Committee	1.50	20.00	20.00
Science Fair	100.00	100.00	100.00
Sciences Calendar	10.00	50.00	50.00
Archivist	—	75.00	75.00
	$7608.06	$8145.00	$8145.00
Estimated Deficit			$ 295.00

A letter from J. A. Stevenson was read in which he submitted his resignation as an Elected Member of the Board of Managers because he is at present Archivist of the Academy. The Board accepted the resignation and in Mr. Stevenson's place appointed Milton Harris.

The Board appointed C. F. W. Muesebeck to fill the vacancy created by the resignation of H. P. Barss as an Elected Member of the Board of Managers.

The meeting adjourned at 8:55 P.M.

F. M. DEFANDORF, *Secretary*